Question 1 (Power Series, 25 points)**.** *What are the first five terms (i.e. the x^0 through x^4 terms) of the power series for $(1-2x)^{3/2}$ about $x=0$?*

Using Newton's Formula we have that

$$
\begin{aligned}
&(1-2x)^{3/2}\\
=&1+(3/2)(-2x)+(3/2)(1/2)(-2x)^2/2+(3/2)(1/2)(-1/2)(-2x)^3/6+(3/2)(1/2)(-1/2)(-3/2)(-2x)^4/24+\ldots\\
=&1-3x+(3/2)x^2+(1/2)x^3+(3/8)x^4+\ldots
\end{aligned}
$$

Question 2 (Subset Counting, 25 points)**.** *How many subsets of* $\{1, 2, \ldots, 7\}$ *either:*

- *Contain* $1, 2,$ *and* $3,$
- *Contain* $3, 4,$ *and* $5,$ *OR*
- *Contain exactly five elements?*

Let A be the set of subsets containing $1, 2, 3,$ B the set of subsets containing $3, 4, 5,$ and C the set of subsets with 5 elements. By Inclusion-Exclusion, we have that

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|.$$

We have $|A| = 2^4 = 16,$ $|B| = 2^4 = 16,$ $|C| = \binom{7}{5} = 21,$ $|A \cap B| = 2^2 = 4,$ $|B \cap C| = \binom{4}{2} = 6,$ $|C \cap A| = \binom{4}{2} = 6,$ $|A \cap B \cap C| = 1.$ Therefore the answer is

$$16 + 16 + 21 - 4 - 6 - 6 + 1 = 38.$$

Question 3 (Same Cycle Counting, 25 points)**.** *Let i, j, k be three distinct numbers between* 1 *and n. For how many permutations of $[n]$ is k in the same cycle as i but not the same cycle as j. Justify your answer.*

We note that we get the same answer no matter what i, j, k are, so we may assume that $k = n$. Writing the permutation in canonical cycle notation, we note that k is in the same cycle as i but not the same cycle as j if and only if j comes before k comes before i in canonical cycle notation. To construct such a canonical cycle notation, there are $\binom{n}{3}$ ways to select the three locations for i, j, k (and then j goes on the first of these, k in the second and i in the third), and $(n-3)!$ ways to arrange the other elements. Thus the answer is $\binom{n}{3}(n-3)! = n!/6$.

Question 4 (Partitions with Distinct Part Sizes, 25 points)**.** *Let $p(n,k)$ denote the number of integer partitions of n with exactly k parts of different sizes. For example, $p(6,2) = 6$ due to the partitions $5+1,\ 4+2,\ 4+1+1,\ 3+1+1+1,\ 2+2+1+1,\ 2+1+1+1+1$. Write the generating function $f(x,y) = \sum_{n,k=0}^{\infty} p(n,k)x^n y^k$ as an infinite product. Hint: For each $m \geq 1$ have a term in your product to account for the number of parts in the partition of size equal to m.*

Letting a_i be the number of copies of i in a partition, $p(n,k)$ is the number of sequences of non-negative numbers a_k with $a_1 + 2a_2 + 3a_3 + \ldots = n$ and the number of non-zero a_i's equal to k. Thus,

$$\begin{aligned} f(x,y) &= \sum_{a_i} x^{a_1+2a_2+3a_3+\cdots} y^{(1 \text{ if } a_1>0)+(1 \text{ if } a_2>0)+(1 \text{ if } a_3>0)+\ldots} \\ &= \prod_{n=1}^{\infty} \sum_{a_n=0}^{\infty} x^{na_n} y^{(1 \text{ if } a_n>0)} \\ &= \prod_{n=1}^{\infty} (1 + x^n y + x^{2n} y + x^{3n} y + \ldots) \\ &= \prod_{n=1}^{\infty} \left(1 + \frac{x^n y}{1-x^n}\right). \end{aligned}$$