**Question 1** (Partition Computation, 25 points)**.** *What is* $p_2(1000)$?

The answer is 500. Note that a partition of 1000 into 2 parts consists of a pair of positive integers $a \geq b > 0$ so that $a + b = 1000$. The latter condition holds if and only if $b = 1000 - a$. Thus we need an integer $a$ so that $a \geq 1000 - a > 0$, which is equivalent to saying that $1000 > a \geq 500$. There are exactly 500 such integers.

**Question 2** (Three Digit Counting, 25 points)**.** *How many odd, three digit numbers have all three digits distinct? Note: the first digit of the number cannot be a leading 0.*

The answer is 320. There are 5 options for the last digit (it must be one of $1, 3, 5, 7, 9$). For every such choice, there are 8 possible options for the first digit (it can be any digit except for 0 and whatever the last digit was). For any choice of that, there are 8 options for the middle digit (it can be any digit except for those chosen as the first or last digit). Thus, by the generalized multiplication rule, the answer is $5 \cdot 8 \cdot 8 = 320$.

**Question 3** (Square Cycles, 25 points)**.** *Let $\pi$ be the $n$-cycle permutation $(123\ldots n)$. What are the lengths of the cycles of $\pi^2$? Justify your answer.*

*Hint: The answer will depend on whether $n$ is even or odd.*

If $n$ is odd there is one cycle of length $n$, if $n$ is even, there are two cycles of length $n/2$.

If $n$ is odd, we have that $\pi^2(1) = 3, \pi^2(3) = 5, \ldots, \pi^2(n-2) = n, \pi^2(n) = 2, \pi^2(2) = 4, \ldots, \pi^2(n-3) = n-1, \pi^2(n-1) = 1$. This gives one cycle of length $n$.

If $n$ is even, we have that $\pi^2(1) = 3, \pi^2(3) = 5, \ldots, \pi^2(n-3) = n-1, \pi^2(n-1) = 1$, giving a cycle of length $n/2$. Similarly, $\pi^2(2) = 4, \pi^2(4) = 6, \ldots, \pi^2(n-2) = n, \pi^2(n) = 2$, giving another cycle of length $n/2$.

**Question 4** (Dice Repeats, 25 points)**.** *One hundred and twenty-seven people each roll four six-sided dice and record their results. Prove that at least two of these people got exactly the same results (the same four numbers, but possibly in a different order).*

Note that the possible outcomes here consist of some number of 1s, some number of 2s, etc. This gives us six non-negative integers that add up to four (since there are four total dice). Thus, each outcome corresponds to a weak composition of 4 into 6 parts. The number of such weak compositions is $\binom{4+6-1}{6-1} = 126$. Thus, with 127 people, by the pigeonhole principle, at least two must have rolled the same results.