Math 184 Homework 6

Fall 2022

Question 1 (Binomial Theorem Identity, 20 points). *Notice that* $(1-x^2)^n = (1+x)^n(1-x)^n$. *By expanding both sides using the Binomial Theorem and comparing terms, derive a combinatorial identity.*

Proof. Expanding the left hand side using the binomial theorem, we have $(1-x^2)^n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}x^{2k}$. On the right hand side, expanding using the binomial theorem we have the following:

$$(1-x)^n(1+x)^n = \sum_{i=0}^{n}(-1)^i\binom{n}{i}x^i\sum_{j=0}^{n}\binom{n}{j}x^j$$

$$= \left(\binom{n}{0}x^0 - \binom{n}{1}x + \binom{n}{2}x^2 - \cdots + (-1)^n\binom{n}{n}x^n\right)\left(\binom{n}{0}x^0 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n\right)$$

$$= \left(\binom{n}{0}^2\right)x^0 + \left(-\binom{n}{0}\binom{n}{1} + \binom{n}{1}\binom{n}{0}\right)x + \left(\binom{n}{0}\binom{n}{2} - \binom{n}{1}\binom{n}{1} + \binom{n}{2}\binom{n}{0}\right)x^2 + \cdots$$

$$+ \left(\binom{n}{0}\binom{n}{n} - \binom{n}{1}\binom{n}{n-1} + \cdots + (-1)^n\binom{n}{n}\binom{n}{0}\right)x^n.$$

Thus the coefficient of the x^l term on the right hand side is $\sum_{m=0}^{l}(-1)^m\binom{n}{m}\binom{n}{l-m}$. From our original equation, which tells us that the coefficient of x^l on the left hand side is equal to the coefficient of x^l on the right hand side, we obtain the combinatorial identity

$$\sum_{m=0}^{l}(-1)^m\binom{n}{m}\binom{n}{l-m} = \begin{cases} 0, & l \text{ is odd,} \\ (-1)^k\binom{n}{k}, & l = 2k \text{ is even} \end{cases}.$$

□

Question 2 (Generating Functions for Polynomials, 20 points). *Show that for m a non-negative integer that*

$$\sum_{n=0}^{\infty} n^m x^n = \sum_{k=0}^{m} S(m,k)k!\left(\frac{x^k}{(1-x)^{k+1}}\right).$$

Proof. We know that

$$\frac{1}{1-x} = 1 + x + x^2 + \dots.$$

Taking the derivative of this expression k times gets us that

$$\frac{k!}{(1-x)^{k+1}} = k! + (k+1)_k x + (k+2)_k x^2 + \ldots = \sum_{i=0}^{\infty}(k+i)_k x^i.$$

Plugging this in to the right hand side of the equation, we get that

$$\sum_{k=0}^{m} S(m,k)k!\left(\frac{x^k}{(1-x)^{k+1}}\right) = \sum_{k=0}^{m} S(m,k)\sum_{i=0}^{\infty}(k+i)_k x^{k+i}$$

Now we consider how we get an x^n term. This occurs exactly when $k+i=n$, or for each fixed value of k, it is when $i=n-k$. This gets us a coefficient of

$$\sum_{k=0}^{m} S(m,k)(n)_k$$

which we know from our study of Stirling numbers is equal to n^m. Therefore the right-hand side is equal to

$$\sum_{n=0}^{\infty} n^m x^n$$

which is what we wanted to show. □

Question 3 (Finite Differences, 30 points)**.** *Given a sequence $a(0), a(1), a(2), \ldots,$ define the first finite difference to be another sequence $\Delta a(n) = a(n) - a(n-1)$ for $n \geq 1$. Define the k^{th} finite difference $\Delta^k a(n)$ to be the sequence obtained by applying the finite difference operation to the sequence $a(i)$ a total of k times.*

(a) Given the generating function $A(x) = \sum_{n=0}^{\infty} a(n)x^n$ give a formula for the generating function for its first finite difference $B(x) = \sum_{n=1}^{\infty} \Delta a(n)x^n$. [15 points]

(b) Show that for the sequence $a(n) = n^k$ for some integer $k \geq 0$ that the $(k+1)^{st}$ finite difference $\Delta^{k+1}a(n)$ is 0 for all sufficiently large n. Hint: You may want to use the result from Question 2 above. [15 points]

Proof. (a) First expand using the definition of the first finite difference,

$$B(x) = \sum_{n=1}^{\infty} \Delta a(n)x^n = \sum_{n=1}^{\infty}(a(n) - a(n-1))x^n,$$

and then split the sums and reindex as needed to match the generating series for $A(x)$ as follows:

$$B(x) = \sum_{n=1}^{\infty} a(n)x^n - \sum_{n=1}^{\infty} a(n-1)x^n = A(x) - a(0) - x\sum_{n=0}^{\infty} a(n)x^n = (1-x)A(x) - a(0).$$

(b) Define $B_{k+1}(x) = \sum_{n=1}^{\infty} b_{k+1}(n)x^n = \sum_{n=k+1}^{\infty} \Delta^{k+1}a(n)x^n$. We wish to reveal a recursive structure in $B_{k+1}(x)$ that will lead us to an expression involving $A(x)$.

$$
\begin{aligned}
B_{k+1}(x) &= \sum_{n=k+1}^{\infty} \Delta^{k+1} a(n) x^n \\
&= \sum_{n=k+1}^{\infty} \Delta^{k} a(n) x^n - \sum_{n=k+1}^{\infty} \Delta^{k} a(n-1) x^n \\
&= \sum_{n=k+1}^{\infty} \Delta^{k} a(n) x^n - x\sum_{n=k+1}^{\infty} \Delta^{k} a(n-1) x^{n-1} \\
&= \sum_{n=k}^{\infty} b_k(n) x^n - b_k(k)x^k - x\sum_{n=k}^{\infty} \Delta^{k} a(n) x^{n} \\
&= (1-x)B_k(x) - b_k(k)x^k \\
&= (1-x)\left((1-x)B_{k-1}(x) - b_{k-1}(k-1)x^{k-1}\right) - b_k(k)x^k \\
&= (1-x)\left((1-x)\left((1-x)B_{k-2}(x) - b_{k-2}(k-2)x^{k-2}\right) - b_{k-1}(k-1)x^{k-1}\right) - b_k(k)x^k \\
&\vdots \\
&= (1-x)^k B(x) - S
\end{aligned}
$$

where

$$
\begin{aligned}
S &= \sum_{i=0}^{k} b_{k-i}(k-i)x^{k-i}(1-x)^i = \sum_{i=0}^{k} b_{k-i}(k-i)x^{k-1}\sum_{l=0}^{i}\binom{i}{l}(-1)^{l-i}x^l \\
&= \sum_{i=0}^{k} b_{k-i}(k-i)\sum_{l=0}^{i}\binom{i}{l}(-1)^{l-i}x^{l+k-1}
\end{aligned}
$$

is a finite degree polynomial. Now we expand once more,

$$B_{k+1}(x) = (1-x)^{k+1}A(x) - (1-x)^k a(0) - S,$$

and since $a(n) = n^k$, $a(0) = 0$, so we have

$$B_{k+1}(x) = (1-x)^{k+1}A(x) - S.$$

Next we expand $A(x)$ using Question 2 as follows:

$$B_{k+1}(x) = (1-x)^{k+1}\sum_{i=0}^{k} S(k,i)i!\left(\frac{x^i}{(1-x)^{i+1}}\right) - S.$$

Bringing the power of $(1-x)^{k+1}$ into the summation, we see that $\frac{x^i}{(1-x)^{i+1}}$ becomes $x^i(1-x)^{k-1}$, which is a finite degree polynomial. In fact, we can expand by binomial theorem

$$x^i(1-x)^{k-1} = x^i\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^{k-1-j}x^j = \sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^{k-1-j}x^{j+i}.$$

In all, we have expressed

$$B_{k+1}(x) = \sum_{i=0}^{k} S(k,i)i!x^i(1-x)^{k-1} - S$$

which is a finite degree polynomial as we have shown the finite summation is over a finite degree polynomial and S is also a finite degree polynomial. Thus if we used the expanded forms and grouped terms, the coefficients of the generating series $B_{k+1}(x)$ would be zero for n sufficiently large, which is to say, for $a(n) = n^k$, $\Delta^{k+1}a(n)$ is 0 for all n sufficiently large.

□

Question 4 (Generating Function Calculations, points). *Derive formulas for the following generating functions:*

(a) The sequence a_n is given by the recurrence $a_0 = a_1 = a_2 = 1$ and $a_n = a_{n-2} + a_{n-3}$ for all $n \geq 3$. Give a formula for the generating function $A(x) = \sum_{n=0}^{\infty} a_n x^n$. [10 points]

(b) The sequence b_n is given by the recurrence $b_0 = 1$ and

$$b_n = \begin{cases} b_{n/2} & \text{if } n \text{ is even} \\ -b_{(n-1)/2} & \text{if } n \text{ is odd} \end{cases}$$

for all $n \geq 0$. Show that the generating function $B(x) = \sum_{n=0}^{\infty} b_n x^n$ satisfies $B(x) = (1-x)B(x^2)$. Give a formula for $B(x)$ as an infinite product. [10 points]

(c) The sequence c_n satisfies the recurrence relation $c_n = \frac{2}{n}\sum_{m=0}^{n-1} c_m + n$. Give a differential equation satisfied by the generating function $C(x) = \sum_{n=0}^{\infty} c_n x^n$. [10 points]

Proof. (a) We write

$$\begin{aligned} A(x) &= \sum_{n=0}^{\infty} a_n x^n = 1 + x + x^2 + \sum_{n=3}^{\infty} a_n x^n \\ &= 1 + x + x^2 + \sum_{n=3}^{\infty} (a_{n-2} + a_{n-3}) x^n \\ &= 1 + x + x^2 + x^2 \sum_{n=3}^{\infty} a_{n-2} x^{n-2} + x^3 \sum_{n=3}^{\infty} a_{n-3} x^{n-3} \\ &= 1 + x + x^2 + x^2 \sum_{i=1}^{\infty} a_i x^i + x^3 \sum_{j=0}^{\infty} a_j x^j \\ &= 1 + x + x^2 + x^2(A(x) - 1) + x^3 A(x) \end{aligned}$$

Solving this for $A(x)$ gives

$$A(x) = \frac{1+x}{1-x^2-x^3}$$

(b) We write $B(x^2) = \sum_{n=0}^{\infty} b_n x^{2n}$

so

$$(1-x)B(x^2) = \sum_{n=0}^{\infty} b_n x^{2n} + \sum_{n=0}^{\infty} (-b_n) x^{2n+1}.$$

What is the coefficient on x^m? It depends on whether m is even or odd. If $m = 2k$ is even, its coefficient is b_k, and if $m = 2k+1$ is odd, the coefficient is $-b_k$. But this is exactly b_m, so $(1-x)B(x^2) = B(x)$.

Now we can use this to write

$$\begin{aligned} B(x) &= (1-x)B(x^2) \\ &= (1-x)(1-x^2)B(x^4) \\ &= (1-x)(1-x^2)(1-x^4)B(x^8) \\ &= \vdots \\ &= \prod_{j=0}^{\infty} (1 - x^{2^j}) \end{aligned}$$

(c) We write

$$\begin{aligned}
C'(x) &= \sum_{n=1}^{\infty} nc_n x^{n-1} \\
&= \sum_{n=1}^{\infty} \left(\frac{2}{n} \sum_{m=0}^{n-1} c_m + n \right) nx^{n-1} \\
&= \sum_{n=1}^{\infty} \left(2 \sum_{m=0}^{n-1} c_m \right) x^{n-1} + \sum_{n=1}^{\infty} n^2 x^{n-1}
\end{aligned}$$

Let's focus on the first sum. We compute

$$\begin{aligned}
\sum_{n=1}^{\infty} \left(2 \sum_{m=0}^{n-1} c_m \right) x^{n-1} &= \sum_{m=0}^{\infty} \sum_{n=m+1}^{\infty} 2c_m x^{n-1} \\
&= \sum_{m=0}^{\infty} 2c_m \sum_{n=m+1}^{\infty} x^{n-1} \\
&= \sum_{m=0}^{\infty} 2c_m x^m \sum_{n=m+1}^{\infty} x^{n-(m+1)} \\
&= \sum_{m=0}^{\infty} 2c_m x^m \sum_{i=0}^{\infty} x^i \\
&= \sum_{m=0}^{\infty} 2c_m x^m \frac{1}{1-x} \\
&= \frac{2C(x)}{1-x}
\end{aligned}$$

Now for the second sum, let $I(x) = \sum_{n=1}^{\infty} n^2 x^{n-1}$. Integrating both sides gives $\int I(x)dx = \sum_{n=1}^{\infty} nx^n$. Now divide both sides by x and integrate again to get

$$\int \frac{\int I(x)dx}{x} dx = \sum_{n=1}^{\infty} x^n = \frac{1}{1-x} - 1.$$

This looks rather frightening, but the key is that we can just reverse the process now that we have the right hand side in a manageable form. So we first take the derivative to get

$$\frac{\int I(x)dx}{x} = \frac{1}{(1-x)^2}$$

Then

$$\int I(x)dx = \frac{x}{(1-x)^2}$$

so

$$I(x) = \frac{1+x}{(1-x)^3}.$$

Putting it all together, we get that $C(x)$ satisfies the differential equation

$$C'(x) = \frac{2C(x)}{1-x} + \frac{1+x}{(1-x)^3}$$

If you're curious, the solution to this equation is

$$C(x) = \frac{c_0 - x - 2\log(1-x)}{(1-x)^2}$$

□

Question 5 (Extra credit, 1 point)**.** *Approximately how much time did you spend working on this homework?*