Math 184 Homework 5

Spring 2022

Solution to Homework 5
Zehong Zhao

This homework is due on gradescope Friday May 13th at 11:59pm pacific time. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in LaTeXis recommend though not required.

Question 1 (Semi-Increasing Sequences, 20 points)**.** *Define a sequence* $a_1, a_2, \ldots, a_n$ *to be* semi-increasing *if* $a_j \geq a_i - 1$ *for all* $j > i$*. Define* $sem(n, k)$ *to be the number of semi-increasing sequences of length* n *consisting of integers from* 1 *to* k*. Determine (as a function of* k*) the generating function*

$$F_k(x) = \sum_{n=0}^{\infty} sem(n, k) x^n.$$

Hint: Proceed by induction on k*. Consider the first occurrence (if one exists) of* k *in the semi-increasing sequence counted by* $sem(n, k)$*.*

Proof. Following the hint, we will prove by induction on k that the generating function $F_k(x) = \frac{1}{1-x}\left(\frac{1-x}{1-2x}\right)^{k-1}$. If $k = 1$, since our semi-increasing sequences of length n will only consist of integer 1, there is only one choice. Hence, the generating function in this case will be

$$F_1(x) = \sum_{n=1}^{\infty} x^n = \frac{1}{1-x} \tag{1}$$

Hence, our base case is verified. For inductive step, assume the generating function $F_{k-1}(x)$ is $\frac{1}{1-x}\left(\frac{1-x}{1-2x}\right)^{k-2}$. We want to calculate the generating function $F_k(x)$. We observe that we have the following recursive relation. Suppose we are given any length n semi-increasing sequence consisting of integers from 1 to k, we record the first occurrence of k in the sequence. Then, the numbers in front of k form an semi-increasing sequence consisting of integers from 1 to $k-1$. Moreover, numbers after that k can only be k or $k-1$. If k does not appear, this would simply be a semi-increasing sequence of length n consisting of integers $1, \ldots, k-1$ (counted by $sem(n, k-1)$). Hence, we conclude that

$$sem(n, k) = sem(n, k-1) + \sum_{i=1}^{n} sem(i-1, k-1) 2^{n-i} \tag{2}$$

Therefore,

$$
\begin{aligned}
F_k(x) &= \sum_{n=0}^{\infty} sem(n,k)x^n && (3)\\
&= \sum_{n=0}^{\infty}\left(sem(n,k-1)x^n + \sum_{i=1}^{n} sem(i-1,k-1)2^{n-i}x^n\right) && (4)\\
&= \sum_{n=0}^{\infty}\left(sem(n,k-1)x^n + \sum_{j=0}^{n-1} sem(j,k-1)2^{n-j-1}x^n\right) && (5)\\
&= F_{k-1}(x) + \sum_{n=0}^{\infty}\sum_{j=0}^{n-1} sem(j,k-1)2^{n-j-1}x^n && (6)
\end{aligned}
$$

Using the product rule for infinite series, the double summation above can be rewritten as

$$
\begin{aligned}
F_{k-1}(x) + \sum_{n=0}^{\infty}\sum_{j=0}^{n-1} sem(j,k-1)2^{n-j-1}x^n &= F_{k-1}(x) + \sum_{n=0}^{\infty} x \sum_{j=0}^{n-1} sem(j,k-1)2^{n-1-j}x^{n-1} && (7)\\
&= F_{k-1}(x) + x\left(\sum_{n=0}^{\infty} sem(n,k-1)x^n\right)\left(\sum_{n=1}^{\infty} 2^n x^n\right) && (8)\\
&= F_{k-1}(x) \times \left(\frac{x}{1-2x}+1\right) && (9)\\
&= F_{k-1}(x) \times \left(\frac{1-x}{1-2x}\right) && (10)\\
&= \frac{1}{1-x}\left(\frac{1-x}{1-2x}\right)^{k-1} && \text{(inductive hypothesis)}
\end{aligned}
$$

We can also arrive at this conclusion more directly as follows. As observed above a semi-increasing sequence of length n using the numbers $1, 2, \ldots, k$ is either a semi-increasing sequence using only $1, 2, \ldots, k-1$ or it is a semi-increasing sequence using the numbers $1, 2, \ldots, k-1$ followed by an arbitrary sequence of k's and $k-1$'s that starts with a k. The generating function for the former term is $F_{k-1}(x)$. For the latter, we need to count the number of ways to find (A) a semi-increasing sequence using numbers $1, 2, \ldots, k-1$ and (B) a sequence using $k-1$'s and k's starting with a k so that the sum of their lengths is equal to n. This is asking for the number of ways to find an object of type A and an object of type B the sum of whose sizes (lengths) equals n. Using the combinatorial interpretation of a product of generating functions, the generating function for this is given by the product of the generating function for objects of type A and the generating function for objects of type B. This is

$\mathrm{F}_{k-1}(x)\left(\sum_{n=1}^{\infty} 2^{n-1}x^n\right) = F_{k-1}(x)(x/(1-2x))$.

□

Question 2 (Partition Identity, 20 points)**.** *Use generating functions to prove that* $p(n) - p(n-1)$ *is the number of partitions of* n *into parts of size bigger than* 1.

Proof. Recall that the generating function of integer partition is given by

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k} \tag{11}$$

This is explained by the product interpretation of generating function and notice that

$$\prod_{k=1}^{\infty} \frac{1}{1-x^k} = (1+x+x^2+\ldots)(1+x^2+x^4+\ldots)\ldots \tag{12}$$

Hence, the using the same argument, the generating function for the integer partitions of n into parts of size bigger than 1 is given by

$$\prod_{k=2}^{\infty} \frac{1}{1-x^k} \tag{13}$$

But notice that

$$\begin{aligned}
\prod_{k=2}^{\infty} \frac{1}{1-x^k} &= (1-x)\prod_{k=1}^{\infty} \frac{1}{1-x^k} && (14)\\
&= (1-x)\sum_{n=0}^{\infty} p(n)x^n && (15)\\
&= \sum_{n=0}^{\infty} p(n)x^n - \sum_{n=0}^{\infty} p(n)x^{n+1} && (16)\\
&= \sum_{n=0}^{\infty} p(n)x^n - \sum_{n=1}^{\infty} p(n-1)x^n && \text{(relabel)}\\
&= \sum_{n=0}^{\infty} [p(n)-p(n-1)]x^n && (p(-1)=0)
\end{aligned}$$

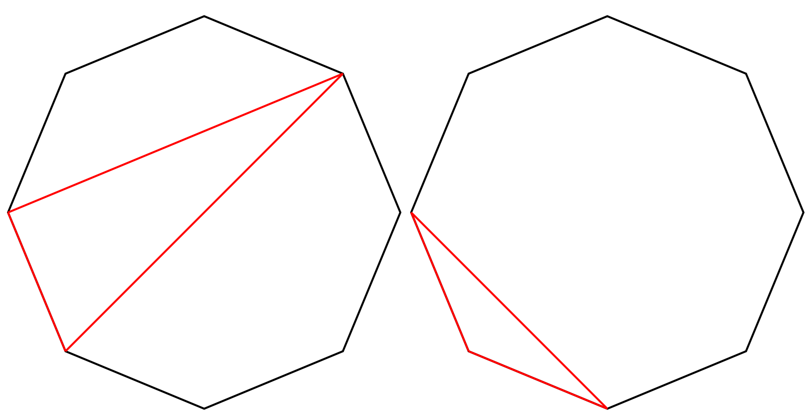
Hence, it follows from the definition of generating function that the number of partitions of n into parts of size bigger than 1 is $p(n) - p(n-1)$. □

Question 3 (Triangulations, 20 points)**.** *Define a* triangulation *of a polygon to be a way of drawing line segments between some pairs of its non-adjacent vertices so that*

1. *No two of these line segments cross (except possibly at endpoints).*
2. *These line segments divide the interior of the polygon into triangles.*

Prove that the number of triangulations of a convex n-gon is the Catalan number C_{n-2}. Hint: Show that it satisfies the recurrence relation for the Catalan numbers.

Proof. Any polygon has to have at least three edges. Denote the number of triangulation of a n-gon by $t(n)$. For a triangle (3-gon) it is clear that there is only one possible triangulation (you don't connect any vertices as all of them are adjacent to each other). Hence, $t(3) = 1$. Suppose we have an $(k+1)$-gon. Let we fixed one of the edge. Then, for any vertices not included in that specific edge, we can construct a unique triangle. The construction is illustrated below with the chosen edge and vertex marked by red:



We have two cases, illustrated in above diagram. If we are in the first case, we have split the $(k+1)$-gon three area: a triangle, a $(i+1)$-gon (to the right of the triangle), and a $(k-i)$-gon, where i is the location of the chosen vertex. Hence, the total number of triangulation in this configuration can be calculated inductively using the product rule:

$$\# = t(i+1)t(k-i) \tag{17}$$

If we are in the second case, the $(k+1)$-gon is splited into 2 area: a triangle and a $(k-1)$-gon. Hence, the number of triangulation in this configuration is

$$\# = t(k-1) \tag{18}$$

With our above discussions, we now prove by induction that $t(n) = C_{n-2}$. The base case is clear as $t(3) = 1 = C_1$. Suppose $t(i)$ is the Catalan number C_{i-2} for all $i < k$. Then, we observe that by our above discussion

$$t(k) = t(k-1) + t(k-1) + \sum_{i=3}^{k-2} t(i)t(k+1-i) \tag{19}$$

$$= C_0 C_{k-3} + C_{k-3}C_0 + \sum_{i=3}^{k-2} C_{i-2}C_{k-1-i} \tag{20}$$

$$= C_0 C_{k-3} + C_{k-3}C_0 + \sum_{j=1}^{k-4} C_j C_{k-3-j} \tag{relabel}$$

$$= \sum_{i=0}^{k-3} C_i C_{k-3-i} \tag{21}$$

$$= C_{k-2} \tag{22}$$

by the recursive definition of Catalan number.

□

Question 4 (Colored Compositions, 40 points)**.** .

(a) *Let a_n be the number of compositions of n (into any number of parts) in which each part in the composition is colored either red or blue. Give a formula for the generating function*

$$\sum_{n=0}^{\infty} a_n x^n.$$

[10 points]

Proof. Here we are going to use the composition rule for generating functions (Theorem 8.17). Let b_n be the structure of coloring the elements of an n-element set by red or blue (to put on the set of intervals) and $G(x)$ be the generating function:

$$G(x) = \sum_{n=0}^{\infty} b_n x^n = \sum_{n=1}^{\infty} 2^n x^n = \frac{2x}{1-2x} \tag{23}$$

Let $F(x)$ be the generating function for the trivial structure (count every n-element set by 1 as we are not putting any structure within any interval), given by

$$F(x) = x + x^2 + \cdots = \frac{x}{1-x} \tag{24}$$

(to use the theorem we need $a_0 = 0$). Hence, the desired generating function is the composition

$$G(F(x)) = \frac{\frac{x}{1-x}}{1 - \frac{2x}{1-x}} = \frac{2x}{1-3x} \tag{25}$$

(This can also be calculated using theorem 8.13, where the structure a_n is choose either red or blue, i.e. $a_n = 2$ for all n.) □

(b) *Using the above generating function, obtain a formula for a_n. [10 points]*

Proof. Then, we observe that

$$\frac{2x}{1-3x} = 2x\sum_{n=0}^{\infty}(3x)^n \tag{26}$$
$$= \sum_{n=0}^{\infty} 2\cdot 3^n x^{n+1} \tag{27}$$
$$= \sum_{n=1}^{\infty} 2\cdot 3^{n-1} x^n \tag{28}$$

Hence, we have

$$\begin{cases} a_0 = 0 \\ a_n = 2\cdot 3^{n-1} \quad n \geq 1 \end{cases} \tag{29}$$

□

(c) *Let b_n be the number of compositions of n (into any number of parts) in which exactly 2 parts are colored blue. Give a formula for the generating function*

$$\sum_{n=0}^{\infty} b_n x^n.$$

[10 points]

Proof. Let b_n be the structure we desired (to put on the set of intervals) and $G(x)$ be the generating function:

$$G(x) = \sum_{n=0}^{\infty}\binom{n}{2}x^n = \sum_{n=0}^{\infty}\frac{x^2}{2}\frac{d^2}{dx^2}(x^n) = \frac{x^2}{2}\frac{d^2}{dx^2}\left(\frac{1}{1-x}\right) = \frac{x^2}{(1-x)^3} \tag{30}$$

Let $F(x)$ be the generating function for the trivial structure (count every n-element set by 1 as we are not putting any structure within any interval), given by

$$F(x) = x + x^2 + \cdots = \frac{x}{1-x} \tag{31}$$

(to use the theorem we need $a_0 = 0$). Hence, the desired generating function is the composition

$$G(F(x)) = \frac{x^2/(1-x)^2}{(1-2x)^3/(1-x)^3} = \frac{x^2(1-x)}{(1-2x)^3} \tag{32}$$

□

(d) *Using the above generating function, obtain a formula for b_n. [10 points]*

Proof. We know that

$$\frac{8}{(1-2x)^3} = \frac{d^2}{dx^2}\frac{1}{1-2x} \implies \frac{1}{(1-2x)^3} = \frac{1}{8}\sum_{n=0}^{\infty}\frac{d^2}{dx^2}(2x)^n = \frac{1}{8}\sum_{n=2}^{\infty} n(n-1)2^n x^{n-2} \tag{33}$$

Therefore,

$$
\begin{aligned}
\frac{x^2(1-x)}{(1-2x)^3} &= (x^2 - x^3)\sum_{n=2}^{\infty}\frac{n(n-1)2^n}{8}x^{n-2} && (34)\\
&= \sum_{n=2}^{\infty}\frac{n(n-1)2^n}{8}x^n - \sum_{n=2}^{\infty}\frac{n(n-1)2^n}{8}x^{n+1} && (35)\\
&= x^2 + \sum_{n=3}^{\infty}\frac{n(n-1)2^n - (n-1)(n-2)2^{n-1}}{8}x^n && (36)
\end{aligned}
$$

Hence, we have

$$
\begin{cases}
a_0 = a_1 = 0\\
a_2 = 1\\
a_n = \frac{(n-1)(n\cdot 2^{n-1}+2^n)}{8} & n \geq 3
\end{cases} \tag{37}
$$

□

Question 5 (Extra credit, 1 point)**.** *Approximately how much time did you spend working on this homework?*