This homework is due on gradescope Friday May 13th at 11:59pm pacific time. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in \LaTeX is recommend though not required.

**Question 1** (Semi-Increasing Sequences, 20 points). Define a sequence $a_1, a_2, \ldots, a_n$ to be semi-increasing if $a_j \geq a_{i} - 1$ for all $j > i$. Define $\text{sem}(n, k)$ to be the number of semi-increasing sequences of length $n$ consisting of integers from 1 to $k$. Determine (as a function of $k$) the generating function

$$F_k(x) = \sum_{n=0}^{\infty} \text{sem}(n, k)x^n.$$

**Hint:** Proceed by induction on $k$. Consider the first occurrence (if one exists) of $k$ in the semi-increasing sequence counted by $\text{sem}(n, k)$.

**Proof.** Following the hint, we will prove by induction on $k$ that the generating function $F_k(x) = \frac{1}{1-x} \left( \frac{1-x}{1-2x} \right)^{k-1}$. If $k = 1$, since our semi-increasing sequences of length $n$ will only consist of integer 1, there is only one choice. Hence, the generating function in this case will be

$$F_1(x) = \sum_{n=1}^{\infty} x^n = \frac{1}{1-x} \quad (1)$$

Hence, our base case is verified. For inductive step, assume the generating function $F_{k-1}(x)$ is $\frac{1}{1-x} \left( \frac{1-x}{1-2x} \right)^{k-2}$. We want to calculate the generating function $F_k(x)$. We observe that we have the following recursive relation. Suppose we are given any length $n$ semi-increasing sequence consisting of integers from 1 to $k$, we record the first occurrence of $k$ in the sequence. Then, the numbers in front of $k$ form an semi-increasing sequence consisting of integers from 1 to $k - 1$. Moreover, numbers after that $k$ can only be $k$ or $k - 1$. If $k$ does not appear, this would simply be a semi-increasing sequence of length $n$ consisting of integers 1, \ldots, $k - 1$ (counted by $\text{sem}(n, k - 1)$). Hence, we conclude that

$$\text{sem}(n, k) = \text{sem}(n, k - 1) + \sum_{i=1}^{n} \text{sem}(i - 1, k - 1)2^{n-i} \quad (2)$$
Therefore,

\[ F_k(x) = \sum_{n=0}^{\infty} \text{sem}(n,k)x^n \]  

\[ = \sum_{n=0}^{\infty} \left( \text{sem}(n,k-1)x^n + \sum_{i=1}^{n} \text{sem}(i-1,k-1)2^{n-i}x^n \right) \]  

\[ = \sum_{n=0}^{\infty} \left( \text{sem}(n,k-1)x^n + \sum_{j=0}^{n-1} \text{sem}(j,k-1)2^{n-j-1}x^n \right) \]  

\[ = F_{k-1}(x) + \sum_{n=0}^{\infty} \sum_{j=0}^{n-1} \text{sem}(j,k-1)2^{n-j-1}x^n \]  

Using the product rule for infinite series, the double summation above can be rewritten as

\[ F_{k-1}(x) + \sum_{n=0}^{\infty} \sum_{j=0}^{n-1} \text{sem}(j,k-1)2^{n-j-1}x^n = F_{k-1}(x) + \sum_{n=0}^{\infty} x \sum_{j=0}^{n-1} \text{sem}(j,k-1)2^{n-j-1}x^{n-1} \]  

\[ = F_{k-1}(x) + x \left( \sum_{n=0}^{\infty} \text{sem}(n,k-1)x^n \right) \left( \sum_{n=1}^{\infty} 2^n x^n \right) \]  

\[ = F_{k-1}(x) \times \left( \frac{x}{1 - 2x} \right) \]  

\[ = \frac{1}{1-x} \left( \frac{1-x}{1-2x} \right)^{k-1} \]  

(Inductive hypothesis)

We can also arrive at this conclusion more directly as follows. As observed above a semi-increasing sequence of length \( n \) using the numbers 1, 2, \ldots, \( k \) is either a semi-increasing sequence using only 1, 2, \ldots, \( k-1 \) or it is a semi-increasing sequence using the numbers 1, 2, \ldots, \( k-1 \) followed by an arbitrary sequence of \( k \)'s and \( k-1 \)'s that starts with a \( k \). The generating function for the former term is \( F_{k-1}(x) \). For the latter, we need to count the number of ways to find \( (A) \) a semi-increasing sequence using numbers 1, 2, \ldots, \( k-1 \) and \( (B) \) a sequence using \( k-1 \)'s and \( k \)'s starting with a \( k \) so that the sum of their lengths is equal to \( n \). This is asking for the number of ways to find an object of type A and an object of type B the sum of whose sizes (lengths) equals \( n \). Using the combinatorial interpretation of a product of generating functions, the generating function for this is given by the product of the generating function for objects of type A and the generating function for objects of type B. This is

\[ F_{k-1}(x) \left( \sum_{n=1}^{\infty} 2^{n-1}x^n \right) = F_{k-1}(x)(x/(1 - 2x)). \]

\[ \square \]

**Question 2** (Partition Identity, 20 points). *Use generating functions to prove that \( p(n) - p(n - 1) \) is the number of partitions of \( n \) into parts of size bigger than 1.*

**Proof.** Recall that the generating function of integer partition is given by

\[ \sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k} \]  

(11)

This is explained by the product interpretation of generating function and notice that

\[ \prod_{k=1}^{\infty} \frac{1}{1-x^k} = (1 + x + x^2 + \ldots)(1 + x^2 + x^4 + \ldots) \ldots \]  

(12)
Hence, the using the same argument, the generating function for the integer partitions of \( n \) into parts of size bigger than 1 is given by

\[
\prod_{k=2}^{\infty} \frac{1}{1-x^k}
\]

But notice that

\[
\prod_{k=2}^{\infty} \frac{1}{1-x^k} = (1 - x) \prod_{k=1}^{\infty} \frac{1}{1-x^k}
\]

\[
= (1 - x) \sum_{n=0}^{\infty} p(n)x^n
\]

\[
= \sum_{n=0}^{\infty} p(n)x^n - \sum_{n=0}^{\infty} p(n)x^{n+1}
\]

\[
= \sum_{n=0}^{\infty} p(n)x^n - \sum_{n=1}^{\infty} p(n-1)x^n
\]

\[
= \sum_{n=0}^{\infty} [p(n) - p(n-1)]x^n
\]

\[
(p(-1) = 0)
\]

Hence, it follows from the definition of generating function that the number of partitions of \( n \) into parts of size bigger than 1 is \( p(n) - p(n-1) \).

**Question 3** (Triangulations, 20 points). Define a triangulation of a polygon to be a way of drawing line segments between some pairs of its non-adjacent vertices so that

1. No two of these line segments cross (except possibly at endpoints).

2. These line segments divide the interior of the polygon into triangles.

Prove that the number of triangulations of a convex \( n \)-gon is the Catalan number \( C_{n-2} \). Hint: Show that it satisfies the recurrence relation for the Catalan numbers.

**Proof.** Any polygon has to have at least three edges. Denote the number of triangulation of a \( n \)-gon by \( t(n) \). For a triangle (3-gon) it is clear that there is only one possible triangulation (you don’t connect any vertices as all of them are adjacent to each other). Hence, \( t(3) = 1 \). Suppose we have an \((k+1)\)-gon. Let we fixed one of the edge. Then, for any vertices not included in that specific edge, we can construct a unique triangle. The construction is illustrated below with the chosen edge and vertex marked by red:

![Triangulation Diagram]

We have two cases, illustrated in above diagram. If we are in the first case, we have split the \((k+1)\)-gon three area: a triangle, a \((i+1)\)-gon (to the right of the triangle), and a \((k-i)\)-gon, where \( i \) is the location of the
chosen vertex. Hence, the total number of triangulation in this configuration can be calculated inductively using the product rule:

\[
\# = t(i + 1)t(k - i)
\]  

(17)

If we are in the second case, the \((k + 1)\)-gon is split into 2 areas: a triangle and a \((k - 1)\)-gon. Hence, the number of triangulation in this configuration is

\[
\# = t(k - 1)
\]  

(18)

With our above discussions, we now prove by induction that \(t(n) = C_{n-2}\). The base case is clear as \(t(3) = 1 = C_{1}\). Suppose \(t(i)\) is the Catalan number \(C_{i-2}\) for all \(i < k\). Then, we observe that by our above discussion

\[
t(k) = t(k - 1) + \sum_{i=3}^{k-2} t(i)t(k + 1 - i)
\]  

(19)

\[
= C_0C_{k-3} + C_{k-3}C_0 + \sum_{i=3}^{k-1} C_{i-2}C_{k-1-i}
\]  

(20)

\[
= C_0C_{k-3} + C_{k-3}C_0 + \sum_{j=1}^{k-4} C_jC_{k-3-j}
\]

(relabel)

\[
= \sum_{i=0}^{k-3} C_iC_{k-3-i}
\]  

(21)

\[
= C_{k-2}
\]  

(22)

by the recursive definition of Catalan number.

\[\Box\]

**Question 4** (Colored Compositions, 40 points).

\(a\) Let \(a_n\) be the number of compositions of \(n\) (into any number of parts) in which each part in the composition is colored either red or blue. Give a formula for the generating function

\[
\sum_{n=0}^{\infty} a_n x^n.
\]

[10 points]

**Proof.** Here we are going to use the composition rule for generating functions (Theorem 8.17). Let \(b_n\) be the structure of coloring the elements of an \(n\)-element set by red or blue (to put on the set of intervals) and \(G(x)\) be the generating function:

\[
G(x) = \sum_{n=0}^{\infty} b_n x^n = \sum_{n=1}^{\infty} 2^n x^n = \frac{2x}{1 - 2x}
\]  

(23)

Let \(F(x)\) be the generating function for the trivial structure (count every \(n\)-element set by 1 as we are not putting any structure within any interval), given by

\[
F(x) = x + x^2 + \cdots = \frac{x}{1 - x}
\]  

(24)

(to use the theorem we need \(a_0 = 0\)). Hence, the desired generating function is the composition

\[
G(F(x)) = \frac{\frac{x}{1-x}}{\frac{1-2x}{1-x}} = \frac{2x}{1 - 3x}
\]  

(25)

(This can also be calculated using theorem 8.13, where the structure \(a_n\) is choose either red or blue, i.e. \(a_n = 2\) for all \(n\).)

\[\Box\]
(b) Using the above generating function, obtain a formula for \( a_n \). [10 points]

Proof. Then, we observe that

\[
\frac{2x}{1-3x} = 2x \sum_{n=0}^{\infty} (3x)^n = \sum_{n=0}^{\infty} 2 \cdot 3^n x^{n+1} = \sum_{n=1}^{\infty} 2 \cdot 3^{n-1} x^n
\]

Hence, we have

\[
\begin{cases}
    a_0 = 0 \\
    a_n = 2 \cdot 3^{n-1} \quad n \geq 1
\end{cases}
\]

(c) Let \( b_n \) be the number of compositions of \( n \) (into any number of parts) in which exactly 2 parts are colored blue. Give a formula for the generating function

\[
\sum_{n=0}^{\infty} b_n x^n.
\]

[10 points]

Proof. Let \( b_n \) be the structure we desired (to put on the set of intervals) and \( G(x) \) be the generating function:

\[
G(x) = \sum_{n=0}^{\infty} \binom{n}{2} x^n = \sum_{n=0}^{\infty} \frac{x^2}{2} \frac{d^2}{dx^2} (x^n) = \frac{x^2}{2} \frac{d^2}{dx^2} \left( \frac{1}{1-x} \right) = \frac{x^2}{(1-x)^3}
\]

Let \( F(x) \) be the generating function for the trivial structure (count every \( n \)-element set by 1 as we are not putting any structure within any interval), given by

\[
F(x) = x + x^2 + \cdots = \frac{x}{1-x}
\]

(to use the theorem we need \( a_0 = 0 \)). Hence, the desired generating function is the composition

\[
G(F(x)) = \frac{x^2/(1-x)^2}{(1-2x)^3/(1-x)^3} = \frac{x^2(1-x)}{(1-2x)^3}
\]

(d) Using the above generating function, obtain a formula for \( b_n \). [10 points]

Proof. We know that

\[
\frac{8}{(1-2x)^3} = \frac{d^2}{dx^2} \frac{1}{1-2x} \implies \frac{1}{(1-2x)^3} = \frac{1}{8} \sum_{n=0}^{\infty} \frac{d^2}{dx^2} (2x)^n = \frac{1}{8} \sum_{n=2}^{\infty} n(n-1)2^n x^{n-2}
\]
Therefore,

\[
\frac{x^2(1-x)}{(1-2x)^4} = (x^2 - x^3) \sum_{n=2}^{\infty} \frac{n(n-1)2^n}{8} x^{n-2} = \sum_{n=2}^{\infty} \frac{n(n-1)2^n}{8} x^n - \sum_{n=2}^{\infty} \frac{n(n-1)2^n}{8} x^{n+1} = x^2 + \sum_{n=3}^{\infty} \frac{n(n-1)2^n - (n-1)(n-2)2^{n-1}}{8} x^n
\]

Hence, we have

\[
\begin{align*}
a_0 &= 0 \\
a_1 &= 0 \\
a_2 &= 1 \\
a_n &= \frac{(n-1)(n-2)2^{n-1} + 2^n}{8} & n \geq 3
\end{align*}
\]

\[
\text{Question 5 (Extra credit, 1 point). Approximately how much time did you spend working on this homework?}
\]

\[
6
\]