Math 184 Homework Solution

Spring 2022

Solution to Homework 3
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This homework is due on gradescope Friday April 22nd at 11:59pm pacific time. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in LaTeXis recommend though not required.

Question 1 (Matchings and Permutations, 20 points)**.** .

(a) *How many permutations of* $[2n]$ *consist of* n *cycles of length 2? [5 points]*

Proof. This is a standard cycle type counting calculation. We want to count the number of permutations of $[2n]$ of cycle type $2 \ldots 2$. Hence, the number will be

$$\frac{(2n!)}{2^n n!} \tag{1}$$

□

(b) *Show a bijection between such permutations and matchings of* $[2n]$*. [5 points]*

Proof. Given such a permutation as above, it can be written as

$$(a_1\ a_2)(a_3\ a_4)\ldots(a_{2n-1}\ a_{2n}) \tag{2}$$

Therefore, we could simply bijectively map this permutation to the matching of a_1 to a_2, a_3 to a_4, etc. This is clearly a bijection (with inverse making each pair of matching into a cycle). □

(c) *How many permutations of* $[3n]$ *consist of* n *cycles of length 3? [5 points]*

Proof. Using the counting formula for permutations of the cycle type $3 \ldots 3$, the number will be

$$\frac{(3n)!}{3^n n!} \tag{3}$$

□

(d) *Does the answer in part c equal to the number of partitions of* $[3n]$ *into sets of size* 3*? Why or why not? [5 points]*

Proof. No it does not. Each such permutation *does* partition $[3n]$ into n cycles of length 3 which gives a partition into sets of size 3. However, the permutation also arranges these triples of elements into a cycle. Since 3 elements can be arranged into a cycle in 2 different ways (for example $1, 2, 3$ can be arranged as (123) or (132)), this means that there are more such permutations than set partitions. □

Question 2 (Square Permutations, 20 points)**.** *Let* π *be a permutation of* $[n]$*. Show that there exists a permutation* σ *with* $\pi = \sigma^2$ *if and only if* π *has a even number of cycles of length* k *for every even number* k*.*

Proof. Suppose $\pi = \sigma^2$ for some permutation. Since distinct cycles commute with each other, it suffices to investigate the effect of squaring cycles of σ. Let $(a_1\ a_2\ \dots\ a_k)$ be a cycle in the cycle decomposition of σ. Observe that if k is odd, we have

$$(a_1\ \dots\ a_k)^2 = (a_1\ a_3\ \dots\ a_k\ a_2\ a_4\ \dots\ a_{k-1}) \tag{4}$$

Therefore, squaring a cycle of odd length gives you a cycle with same length. If k is even, we have

$$(a_1\ \dots\ a_k)^2 = (a_1\ a_3\ \dots\ a_{k-1})(a_2\ a_4\ \dots\ a_k) \tag{5}$$

Summarizing the above result, it is clear that the only way to obtain an even cycle is using case 2. In that case, it is clear that every even cycle comes with pairs. Hence, there can only be even numbers of cycles of even length.

Suppose π has a even number of cycles of length k for every even number k. We first need to show that every cycle τ of odd length can be written as a square. Suppose τ has length k. Then, it is clear that $\tau^k = 1$. Therefore, $\tau = \tau^{k+1}$. Since k is odd, $k+1$ is even. Hence, we have $\tau = (\tau^{\frac{k+1}{2}})^2$, which is a square. Suppose we are given two cycles $\tau_1 = (a_1\ \dots\ a_k)$ and $\tau_2 = (b_1\ \dots\ b_k)$ of even length k in the cycle decomposition, they must be distinct by assumption of cycle decomposition. Then, we have

$$\tau_1\tau_2 = (a_1\ b_1\ a_2\ b_2\ \dots\ a_k\ b_k)^2 \tag{6}$$

by direct computation.

□

Question 3 (Largest in its Cycle, 30 points)**.** *For integers* $1 \le k \le n$*, how many permutations of* $[n]$ *have* k *as the largest element in its cycle? Hint: Consider the canonical cycle representation of such permutations.*

Proof by Haixiao. Let $\pi \in S_n$, where S_n is the set of all permutations on $[n]$, be a permutation of its canonical cycle notation(CCN), satisfying the requirements that k is the largest element in its cycle, which can be written as

$$(\cdots)\cdots(k\cdots)\cdots(\cdots) \tag{7}$$

By the requirements of CCN, the cycles are arranged in the increasing order of their largest elements. Consequently, if $j > k$, then j should only appear in cycles of π that are right to k. Let $g(\pi)$ be the permutation obtained from π by removing the parentheses and reading the entries as a permutation in the one-line notation. For example, $\pi = (412)(53)$, then $g(\pi) = g((412)(53)) = 41253$. By **Transition Lemma** [Bona and Stanley(2017), Lemma 6.15], g is a bijection from the set S_n onto S_n. Therefore, it is enough to count the number of one-line permutations where j are right to k if $j \in \{k+1, k+2, \cdots, n\}$. We first choose $n-k+1$ positions from n, where $\binom{n}{n-k+1}$ choices available. Then we put $\{k, k+1, \cdots, n-1, n\}$ to those $n-k+1$ positions, with the only constraint that k takes the first position, and there are in total $(n-k)!$ ways to arrange them in the desired order. Finally, we put the numbers $\{1, 2, \cdots, k-1\}$ into the $k-1$ positions left, where we have $(k-1)!$ ways to arrange them. Therefore, the total number of the desired permutations should be

$$\binom{n}{n-k+1}(n-k)!(k-1)! = \frac{n!}{(n-k+1)!(k-1)!}(n-k)!(k-1)! = \frac{n!}{n-k+1}. \tag{8}$$

□

Alternative proof:

Proof. Let π be a permutation of $[n]$. From the definition of the canonical cycle presentation, k is the largest element in the permutation if and only if the numbers before k in the canonical cycle presentation are less than k. Therefore, we count the number of permutation in the alternative description.

Define the following bijection $S_n \to [n] \times [n-1] \times \cdots \times [1]$, where π is sent to $(a_1, \dots, a_n)$ with a_i is the number of j such that $i \leq j$ and j is not to the left of i (in the canonical cycle presentation). For example, $(1)(423)(65)$ is sent to $(6, 4, 3, 3, 1, 1)$. Then, based on our criterion, the permutations we want are precisely those whose image under the bijection has kth entry $n - k + 1$ (all element that are larger than k is to the right). Hence, since there are in total $n - k + 1$ possibility of image of the kth entry, it is clear that the number of such permutation is

$$\frac{n!}{n-k+1} \tag{9}$$

□

Question 4 (Sterling Number Bounds, 30 points). *Show the following size bounds on Sterling numbers of the first kind for $1 \leq k \leq n$:*

(a) $c(n,k) \geq (n-1)!/(k-1)!$ *[10 points]*

Proof. We note that $c(n,k)$ is at least the number of permutations of $[n]$ with one cycle of length $n-k+1$ and $k-1$ cycles of length 1. If $k = n$, there is only one such cycle. Otherwise, using our formula for counting the number of permutations with a given cycle structure we have that

$$c(n,k) \geq \frac{n!}{(n-k+1)1^{k-1}(k-1)!1!} \geq \frac{n!}{n(k-1)!} = \frac{(n-1)!}{(k-1)!}.$$

□

Alternative proof:

Proof. Here we use the formula in the hint. Translation: $c(n,k)$ is the coefficient of x^k of the polynomial $x(x+1)\dots(x+n-1)$. To get the degree k coefficients, we need to sum all choice of $n-k$ numbers in the product. But to get a bound, we can sum a subset of all possible choice. We choose to combine x's from the term $x, x+1, \dots, x+k-1$ and multiply the numbers from the term $x+k, x+k+1, \dots, x+n-1$. By multiplying these out, we get a contribution

$$(n-1)(n-2)\dots k x^k = \frac{(n-1)!}{(k-1)!}x^k \tag{10}$$

Since this is only one of the combination, we conclude that

$$c(n,k) \geq \frac{(n-1)!}{(k-1)!} \tag{11}$$

□

(b) For any positive integer a,

$$c(n,k) \leq n!\binom{n+a}{a}/(a+1)^k.$$

Hint: you may want to use the relation that

$$\sum_{k=1}^{n} x^k c(n,k) = x(x+1)\cdots(x+n-1).$$

[20 points]

Proof. Using the hint,

$$(a+1)^k c(n,k) \leq \sum_{k=1}^{n} (a+1)^k c(n,k) = (a+1)(a+2)\dots(a+n) = n!\binom{n+a}{a} \tag{12}$$

Therefore, we get the desired inequality by dividing $(a+1)^k$ on both side. □

Question 5 (Extra credit, 1 point). *Approximately how much time did you spend working on this homework?*

References

[Bona and Stanley(2017)] Miklos Bona and Richard P Stanley. *A walk through combinatorics: an introduction to enumeration and graph theory.* WORLD SCIENTIFIC, New Jersey, fourth edition, 2017.