Math 184 Homework Solution

Spring 2022

Solution to Homework 2
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This homework is due on gradescope Friday April 15th at 11:59pm pacific time. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in LaTeXis recommend though not required.

Question 1 (Composition Bijections, 30 points). .

(a) *Give a bijection between the set of compositions of n into parts of size 1 and 2 and the set of compositions of $n+2$ into parts of size at least 2. [15 points]*

(b) *Give a bijection between the set of weak compositions of n into $k+1$ parts and the set of weak compositions of k into $n+1$ parts. Hint: use stars and bars. [15 points]*

Proof. (a): Let X denote the set of composition of n into parts of size 1 or 2, and Y denote the set of composition of $n+2$ into parts of size at least 2. We first define a function $f : X \to Y$ as follows: Given an element $x \in X$, we can write it as

$$(a_1, a_2, \ldots, a_k) \text{ with } \sum_{i=1}^{k} a_i = n \text{ and } a_i \in \{1, 2\}, \forall i.$$

We append an auxiliary entry $a_0 := 2$ to the front of x, so we obtain $x' = (a_0, a_1, \ldots, a_k)$. The basic idea of our construction is to replace the sequence $2 + 1 + 1 + 1 + 1 + \ldots + 1$ in the composition into parts of size 1 and 2 with $k+2$ (where there were k 1's in the first sum).

More specifically, we consider the indices $0 = i_1 < i_2 < \cdots < i_l \leq k$ that satisfies

$$a_{i_1} = a_{i_2} = \cdots = a_{i_l} = 2 \text{ and } a_i = 1 \text{ for any } i \notin \{a_{i_1}, \ldots, a_{i_l}\}.$$

In words, a_{i_j} are those entries of x' that equals 2. For the sake of presentation, we define $i_{l+1} = k+1$. We now define, for all $j = 1, 2, \ldots, l$,

$$b_j = \sum_{i=i_j}^{i_{j+1}-1} a_i.$$

Clearly, we have

$$\sum_{j=1}^{l} b_j = \sum_{j=1}^{l} \sum_{i=i_j}^{i_{j+1}-1} a_i = \sum_{i=0}^{k} a_i = a_0 + \sum_{i=1}^{k} a_i = n + 2.$$

This means $y = (b_1, b_2, \ldots, b_l)$ is a composition of $n+2$. Also, we know that $b_j \geq a_{i_j} = 2$, so $y \in Y$. We define our function f by taking $f(x) = y$.

Now we show that f is a bijection, by constructing its inverse $g : Y \to X$ as follows: Given an element $y \in Y$, we can write it as

$$(b_1, b_2, \ldots, b_l) \text{ with } \sum_{j=1}^{l} b_j = n + 2 \text{ and } b_j \geq 2, \forall j.$$

For each $j = 1, \dots, l$ we define a small composition x_j for b_j by setting

$$x_j = (2, 1, \dots, 1) \text{ where 1 repeats } b_j - 2 \text{ times.}$$

Now we take $x' = (a_0, a_1, \dots, a_k)$ to be the sequence obtained by connecting $x_1, x_2, \dots, x_l$ altogether in a row. Clearly, $a_0 = 2$ as it's the first entry of x_1 and each entry of x' is either 1 or 2. We also have

$$\sum_{i=1}^{k} a_i = \sum_{j=1}^{l} b_j = n + 2.$$

So if we take x to be $(a_1, \dots, a_k)$, x is a composition of $n + 2 - 2 = n$. This means $x \in X$ and we define g by taking $g(y) = x$.

Finally, based on how we constructed f and g, if $f(x) = y$, we must have $g(y) = x$ and vice versa. We conclude that f is a bijection.

(b): Let X be the set of weak compositions of n into $k+1$ parts and X' be the set of sequences of n stars and k bars. Similarly, let Y be the set of composition of k into $n+1$ parts and Y' be the set of sequences of k stars and n bars. We learned in class (or see the proof of Theorem 5.2 in textbook, where the author used balls versus walls instead of stars versus bars) that there's a bijection between X and X' and there's a bijection between Y and Y'. To show there's a bijection between X and Y, it suffices for us to find a bijection $f : X' \to Y'$.

We can define the f as follows: Given a sequences, denoted by x, of n stars and k bars, we consider a new sequence y of k stars and n bars, where each star of x is changed to a bar and each bar of x is changed to a star. We can define f by letting $f(x) = y$. This function f is a bijection since we can construct its inverse $g : Y' \to X'$ similarly: Given a sequence y of k stars and n bars, we switch the role of stars and bars and define the new sequence x as the image of y under g. □

Question 2 (Sterling Number Inequalities, 25 points). *Prove that for all $n \geq k > 0$ that*

$$k^{n-k} \leq S(n, k) \leq k^n / k!$$

Hint: Relate $S(n, k)$ to the number of functions from $\{1, 2, \dots, n\}$ to $\{1, 2, \dots, k\}$.

Proof. Let N be the number of surjective functions from $\{1, 2, \dots, n\}$ to $\{1, 2, \dots, k\}$. We have the identity

$$N = k! \cdot S_{n,k}.$$

We give a justification to this identity. (It's also proved in textbook as Corollary 5.9.) Consider the process to construct an arbitrary surjective function from $\{1, 2, \dots, n\}$ to $\{1, 2, \dots, k\}$. First we partition $\{1, 2, \dots, n\}$ into k parts $P_1, P_2, \dots, P_k$. There are $S(n, k)$ ways to do this; Then we take a permutation σ of $\{1, 2, \dots, k\}$. There are $k!$ ways to do this; Finally, we define our surjective function to map every element in each P_i to $\sigma(i)$. There's only one way to do this. The claimed identity follows from the principle of multiplication.

There are only k^n functions from $\{1, 2, \dots, n\}$ to $\{1, 2, \dots, k\}$ in total, because for each element in $\{1, 2, \dots, n\}$, there are k options for its image. So $N \leq k^n$ and by the previous identity, we have $S_{n,k} \leq k^n / k!$ as wanted.

There are at least $k! k^{n-k}$ surjective functions from $\{1, 2, \dots, n\}$ to $\{1, 2, \dots, k\}$, because we can explicitly construct this many according to the following procedure: First, we choose a permutation σ of $\{1, 2, \dots, k\}$. There are $k!$ ways to do this; Then, for each $i \in \{k+1, k+2, \dots, n\}$, we choose a number $a_i \in \{1, 2, \dots, k\}$. There are k^{n-k} ways to do this; Finally, we define our surjective function by mapping each $i \in \{1, 2, \dots, k\}$ to $\sigma(i)$ and mapping each $i \in \{k+1, k+2, \dots, n\}$ to a_i. Note that such a function is surjective as each element in $\{1, 2, \dots, k\}$ is an image of σ. Therefore, we have $N \geq k! k^{n-k}$ and by the identity in first paragraph, we have $S(n, k) \geq k^{n-k}$ as wanted. □

Question 3 (Partition Identity, 45 points). *Prove that:*

$$p(n) = \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} \sum_{m=0}^{n-k^2} p_{\leq k}(m) p_{\leq k}(n - k^2 - m).$$

Where $p_{\leq k}(n)$ denotes the number of partitions of n with at most k parts. Hint: Count the number of partitions with a $k \times k$ box in the upper left of the Ferrers diagram.

Proof. Let $F(n)$ be the set of Ferrers diagrams of size n and $F_{\leq k}(n)$ be the set of Ferrers diagrams of size n and at most k rows. From the lecture we know that $p(n) = |F(n)|$ and $p_{\leq k}(n) = |F_{\leq k}(n)|$.

We define $F(n, k)$ to be the set of Ferrers diagrams of size n such that the largest square grid of boxes in the upper left corner is of size $k \times k$. See Figure 1 for an example. Since every Ferrers diagram has its upper left corner in some square grid and such a square grid is unique given it's the largest, we have $F(n) = \bigcup_{i=1}^{\infty} F(n, k)$ and $F(n, k_1) \cap F(n, k_2) = \emptyset$ whenever $k_1 \neq k_2$.

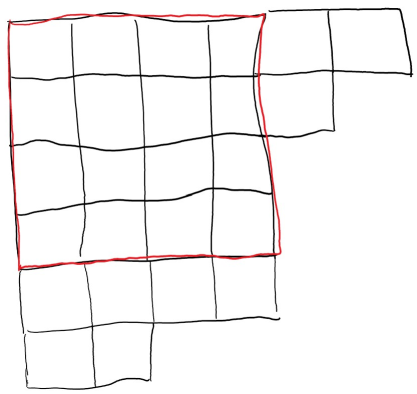

Figure 1: The diagram belongs to $F(25, 4)$, the largest square grid of boxes in the upper left corner is marked red.

If $k > \lfloor\sqrt{n}\rfloor$, a $k \times k$ grid of boxes contains at least $(\lfloor\sqrt{n}\rfloor + 1)^2 > n$ small boxes, so a Ferrers diagram of size n cannot contain a $k \times k$ grid, which implies $F(n, k) = \emptyset$ as well. So we have $F(n) = \bigcup_{k=1}^{\lfloor\sqrt{n}\rfloor} F(n, k)$ and this implies

$$|F(n)| = \sum_{k=1}^{\lfloor\sqrt{n}\rfloor} |F(n, k)|.$$

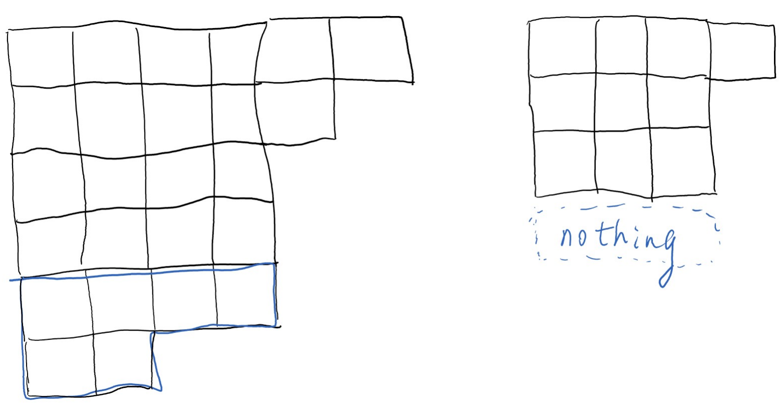

Figure 2: The left diagram belongs to $F(25, 4, 6)$. The right diagram belongs to $F(10, 3, 0)$. The part below the largest square grid in the upper left corner is marked blue.

For each $1 \leq k \leq \lfloor\sqrt{n}\rfloor$, we define $F(n, k, m)$ to be the set of Ferrers diagrams that's in $F(n, k)$ and have m small squares below the largest square grid in the upper left corner. See Figure 2 for examples. We have $F(n, k) = \bigcup_{m=0}^{\infty} F(n, k, m)$ and $F(n, k, m_1) \cap F(n, k, m_2) = \emptyset$ whenever $m_1 \neq m_2$. Notice that any Ferrers diagram in $F(n, k, m)$ contains at least $k^2 + m$ small squares, so $F(n, k, m) = \emptyset$ whenever $m > n - k^2$, and this implies

$$|F(n,k)| = \sum_{m=0}^{n-k^2} |F(n,k,m)|.$$

we now argue that $|F(n,k,m)| = |F_{\leq k}(m)||F_{\leq k}(n-k^2-m)|$. Consider the following procedure to construct an arbitrary element in $F(n,k,m)$: First, choose a Ferrers diagram F_1 of size m that has at most k columns, by conjugation, it's the same as choosing a Ferrers diagram of size m with at most k rows. So there are $|F_{\leq k}(m)|$ options for this; Then choose a Ferrers diagram F_2 of size $n-k^2-m$ with at most k rows. There are $|F_{\leq k}(n-k^2-m)|$ options for this; Finally, take a $k \times k$ grid of small square boxes, put F_1 below the grid and align their leftmost columns, and put F_2 at the right of the grid and align their upmost rows. There's only one way to do this. In this way, we can construct every element in $F(n,k,m)$ exactly once. So, by principle of multiplication, we have the claim identity.

Finally, we combine all the identities we have so far.

$$\begin{aligned}
p(n) = |F(n)| &= \sum_{k=1}^{\lfloor\sqrt{n}\rfloor} |F(n,k)| \\
&= \sum_{k=1}^{\lfloor\sqrt{n}\rfloor} |F(n,k)| = \sum_{m=0}^{n-k^2} |F(n,k,m)| \\
&= \sum_{k=1}^{\lfloor\sqrt{n}\rfloor} |F(n,k)| = \sum_{m=0}^{n-k^2} |F_{\leq k}(m)||F_{\leq k}(n-k^2-m)| \\
&= \sum_{k=1}^{\lfloor\sqrt{n}\rfloor} |F(n,k)| = \sum_{m=0}^{n-k^2} |p_{\leq k}(m)||p_{\leq k}(n-k^2-m)|,
\end{aligned}$$

which is exactly what we wish to prove. □

Question 4 (Extra credit, 1 point)**.** *Approximately how much time did you spend working on this homework?*