Math 184 Homework Solution

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Solution to Homework 1
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This homework is due on gradescope Friday April 8th at 11:59pm pacific time. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in LaTeXis recommend though not required.

Question 1 (Symmetric Polynomials, 50 points)**.** *Call a polynomial P in the variables $x_1, x_2, \ldots, x_n$* symmetric *if switching any of the variables leaves P unchanged. So for example $x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3$ is a symmetric polynomial in x_1, x_2, x_3 but $x_1 + 2x_2 + 3x_3$ is not. A particular example of this are the* power-sum symmetric polynomials *defined as $p_k = \sum_{i=1}^{n} x_i^k$. Show that any symmetric polynomial can be written as a polynomial in the power-sum symmetric polynomials. For example, if $P(x, y, z) = x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3$, then $P = p_2 - p_1^3/6 + p_1p_2/2 - p_3/3$.*

Hint: You will want to use induction, but not on the number of variables. Start with a polynomial P and find a way to add or subtract products of the power-sum polynomials to simplify it. Repeat this until there is nothing left.

Proof. Let $P(x_1, \cdots, x_n)$ be the symmetric polynomial. Since whenever P contains a monomial, it must also have the symmetric monomials with the same coefficient, we can see that P must be a linear combination of terms of the form:

$$m_l = \sum_{\substack{i_1, \cdots, i_l \in [n] \\ i_1, \cdots, i_l \text{ are distinct}}} x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_l}^{\lambda_l}. \tag{1}$$

We claim that each such m_l can be expressed as a polynomial in terms of *power-sum symmetric polynomials* p_k for each $l \in \{1, 2, \cdots, n\}$. We will prove the desired argument by induction on l, i.e., the number of variables appearing in the relevant monomials.

1. $l = 1$, then $m_1 = \sum_{i_1 \in [n]} x_{i_1}^{\lambda_1} = p_{\lambda_1}$.
2. $l = 2$, then

$$m_2 = \sum_{\substack{i_1, i_2 \in [n] \\ i_1 \neq i_2}} x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} = \sum_{i_1 \in [n]} \sum_{i_2 \in [n]} x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} - \sum_{\substack{i_1, i_2 \in [n] \\ i_1 = i_2}} x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \tag{2a}$$

$$= (\sum_{i_1 \in [n]} x_{i_1}^{\lambda_1})(\sum_{i_2 \in [n]} x_{i_2}^{\lambda_2}) - \sum_{i_1 \in [n]} x_{i_1}^{\lambda_1 + \lambda_2} \tag{2b}$$

$$= p_{\lambda_1} p_{\lambda_2} - p_{\lambda_1 + \lambda_2} \tag{2c}$$

   Note that second term in Equation (2b) is a monomial with only 1 distinct variables.

3. Suppose that for $l = t$, we can express

$$m_t = \sum_{i_1, \cdots, i_t \in [n]} x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_t}^{\lambda_t}, \quad \text{distinct } i_1, \cdots, i_t, \tag{3a}$$

by a polynomial in terms of $p_{\lambda_1}, \cdots, p_{\lambda_t}$. As for case $l = t+1$, we expand $p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_{t+1}}$, then

$$
\begin{aligned}
p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_{t+1}} &= \left(\sum_{i_1 \in [n]} x_{i_1}^{\lambda_1}\right)\left(\sum_{i_2 \in [n]} x_{i_2}^{\lambda_2}\right) \cdots \left(\sum_{i_t \in [n]} x_{i_t}^{\lambda_t}\right)\left(\sum_{i_{t+1} \in [n]} x_{i_{t+1}}^{\lambda_{t+1}}\right) && \text{(4a)} \\
&= \sum_{i_1, \cdots, i_{t+1} \in [n]} x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_{t+1}}^{\lambda_{t+1}} \quad \text{distinct } i_1, \cdots, i_{t+1} && \text{(4b)} \\
&\quad + (\text{monomials with number of distinct variables } \le t) && \text{(4c)}
\end{aligned}
$$

As we can see, Equation (4c) should only contain terms like m_l with $l \le t$ with various constants in front of each m_l to match coefficients, since some of the indices $i_1, \cdots, i_l$ may coincide with each other, thus the number of distinct variables decreases. Consequently, m_{t+1}, a.k.a. the term in Equation (4b), can be written as

$$
m_{t+1} = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_{t+1}} - \left[C_t m_t + C_{t-1} m_{t-1} + \cdots + C_1 m_1\right], \tag{5}
$$

where $C_t, \cdots, C_1$ are some constants to match the coefficients. By inductive assumption, second term on the right hand side of Equation (5) can be represented by a polynomial with at most t terms from $\{p_{\lambda_1}, p_{\lambda_2}, \cdots, p_{\lambda_{t+1}}\}$, which finishes the proof.

□

Question 2 (Simultaneous Rational Approximation, 20 points)**.** *Dirichlet's Theorem is useful when you want to approximate one number by rationals, but what if you have two? Suppose that you have two real numbers x and y and want to find integers n, k, m so that $|x - n/m|$ and $|y - k/m|$ are both small. Prove that for any integer q, one can always find n, k, m with $|m| \le q^2$ so that $|x - n/m|$ and $|y - k/m|$ are each at most $1/(mq)$.*

Proof. We consider the real numbers lx, ly, and the decomposition $lx = \lfloor lx \rfloor + a_l$, $ly = \lfloor ly \rfloor + b_l$ for $l = 0, 1, 2, \cdots, q^2$, where a_l, b_l denote the decimal parts of lx, ly respectively. Consider the pairs $(a_0, b_0), \cdots, (a_{q^2}, b_{q^2})$, which are in total $q^2 + 1$ pairs. One can divide the unit square $[0, 1) \times [0, 1)$ into q^2 smaller pieces of area $\frac{1}{q^2}$. Now, we have $q^2 + 1$ pairs with q^2 squares. Therefore, by the pigeonhole principle, at least two of them are in the same square. Without loss of generality, we call those pairs (a_i, b_i) and (a_j, b_j) with $i < j$. Now

$$
|a_j - a_i| = |(jx - \lfloor jx \rfloor) - (ix - \lfloor ix \rfloor)| = |(j-i)x - (\lfloor jx \rfloor - \lfloor ix \rfloor)| < \frac{1}{q} \tag{6a}
$$

$$
|b_j - b_i| = |(jy - \lfloor jy \rfloor) - (iy - \lfloor iy \rfloor)| = |(j-i)y - (\lfloor jy \rfloor - \lfloor iy \rfloor)| < \frac{1}{q} \tag{6b}
$$

Let $m = j - i$, $n = \lfloor jx \rfloor - \lfloor ix \rfloor$ and $k = \lfloor jy \rfloor - \lfloor iy \rfloor$. Dividing both sides by m will result in

$$
\left| x - \frac{n}{m} \right| < \frac{1}{mq}, \tag{7a}
$$

$$
\left| y - \frac{k}{m} \right| < \frac{1}{mq}. \tag{7b}
$$

□

Question 3 (Counting Matchings, 30 points)**.** *Let* $[12]$ *denote the set* $\{1, 2, 3, \ldots, 12\}$*. A* matching *of* $[12]$ *is a way of partitioning the elements into pairs so that each element is in exactly one pair. For example, one matching is* $\{1, 3\}, \{2, 7\}, \{4, 10\}, \{5, 6\}, \{8, 11\}, \{9, 12\}$*. For each of the following count the number of matchings with the given property both as a formula and by giving the exact number. Remember to justify your answer.*

(a) The number of all matchings of $[12]$*. [5 points]*

(b) The number of matchings of $[12]$ *where each even number is paired with another even number. [5 points]*

(c) *The number of matchings of* [12] *where each even number is paired with an odd number. [5 points]*

(d) *The number of matchings of* [12] *where each number is paired with another number at most* 2 *away from it (for this you will want to relate the number of such pairings of* $[2n]$ *to the number of such pairings of* $[2(n-1)]$ *and of* $[2(n-2)]$ *and produce a recurrence). [5 points]*

(e) *The number of matchings of* [12] *where each of* 1, 2, 3 *is paired to one of* 1, 2, 3, 4, 5, 6, 7, 8, 9. *[5 points]*

(f) *The number of matchings of* [12] *where there are exactly 2 pairs of even numbers that are matched together. [5 points]*

Proof. (a) We start pairing from 1, and there are 11 options, then the first pair is removed and 10 numbers left. We then find the matching for smallest remaining element, and there are 9 options, then the second pair is removed and 8 elements left. By repeating this process, the number of choices for next iterations should be $7, 5, 3, 1$. Therefore, the total number of matchings is $11 * 9 * 7 * 5 * 3 * 1 = 10395$.

(b) We choose 3 pairs from odd and even separately. Following the same strategy in part (a), the number of matchings from 6 distinct elements should be $5 * 3 * 1 = 15$. The pairings for even numbers and odd numbers are independent from each other, then the number of such type matchings of [12] is $15*15 = 225$.

(c) We consider the matching of each even number from 2 to 12 sequentially. There are 6 options for 2, then the first pair is removed. For the second round, there are 5 options for 4. By repeating this process, the number of choices for next iterations should be $4, 3, 2, 1$. Therefore, the total number of matchings is $6! = 720$.

(d) Let k_n denote the number of such pairings in $[2n]$. We want to find the relationship between k_n, k_{n-1} and k_{n-2} and then establish the recurrence equation. We consider the matching of the last number $2n$. It can be only matched to $2n-1$ or $2n-2$, since it can only be matched at most 2 away from itself. That's why k_{n-3} is not necessary to our consideration. In first case where $2n$ is matched to $2n-1$, $[2n]$ can be treated as $\{2n-1, 2n\}$ being added to $[2(n-1)]$ and there are k_{n-1} options. In the second case where $2n$ is matched to $2n-2$ and $\{2n-3, 2n-1\}$ is a pair automatically, $[2n]$ can be viewed as $\{2n-3, 2n-2, 2n-1, 2n\}$ being added to $[2(n-2)]$. As a result, k_{n-2} options are available there. Consequently, we have $k_n = k_{n-1} + k_{n-2}$ as shown in Figure 1, which is a Fibonacci Sequence. Obviously, we have $k_1 = 1$ and $k_2 = 2$ for the base cases by simple counting. According to the recurrence relation above, we have $k_3 = 3$, $k_4 = 5$, $k_5 = 8$, and finally $k_6 = 13$ which is of our interest.

For readers who are interested in the result for any $n \in \mathbb{N}_+$, we claim that the formula for k_n is

$$k_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \right). \tag{8}$$

Let $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$, then $k_n = \frac{1}{\sqrt{5}}(a^{n+1} - b^{n+1})$. Note that

$$a^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{3+\sqrt{5}}{2} = a+1\,, \tag{9a}$$

$$b^2 = \left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{1-2\sqrt{5}+5}{4} = \frac{3-\sqrt{5}}{2} = b+1\,. \tag{9b}$$

We now start our induction.

(i) $n = 1$, obviously, $k_1 = \frac{1}{\sqrt{5}}(a^2 - b^2) = \frac{1}{\sqrt{5}}[(a+1) - (b+1)] = 1$.

(ii) $n = 2$,

$$k_2 = \frac{1}{\sqrt{5}}(a^3 - b^3) = \frac{1}{\sqrt{5}}[(a+1)a - (b+1)b] = \frac{1}{\sqrt{5}}[(a+b+1)(a-b)] = 2\,. \tag{10}$$

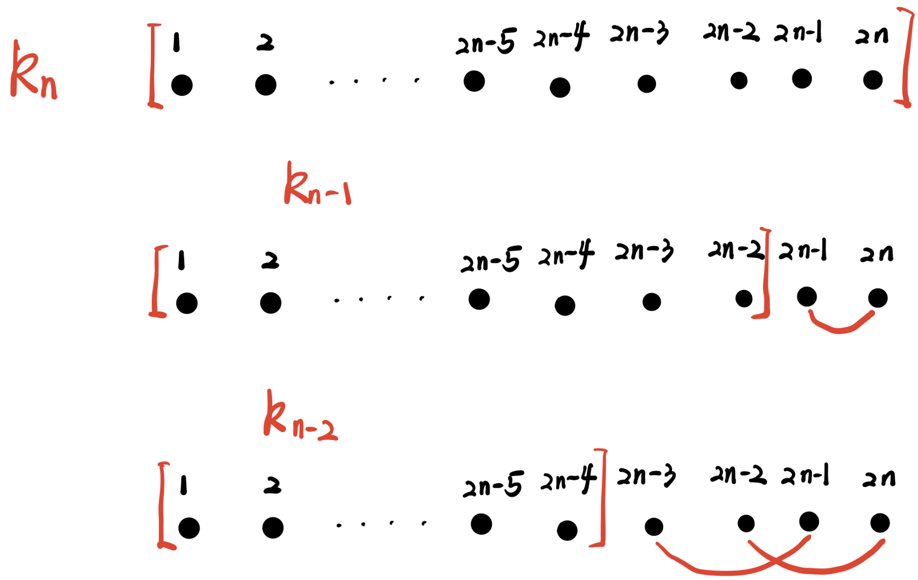

Figure 1: Illustration of $k_n = k_{n-1} + k_{n-2}$.

(iii) For $n = l$, we write

$$
\begin{aligned}
k_l &= \frac{1}{\sqrt{5}}(a^{l+1} - b^{l+1}) = \frac{1}{\sqrt{5}}(a^2 a^{l-1} - b^2 b^{l-1}) && \text{(11a)}\\
&= \frac{1}{\sqrt{5}}[(a+1)a^{l-1} - (b+1)b^{l-1}] = \frac{1}{\sqrt{5}}[(a^l - b^l) + (a^{l-1} - b^{l-1})] && \text{(11b)}\\
&= k_{l-1} + k_{l-2}\,. && \text{(11c)}
\end{aligned}
$$

The proof is finished here since the recurrence formula is finally obtained.

(e) There are 2 cases. First, 2 numbers from $\{1, 2, 3\}$ are matched together ($\binom{3}{2}$ choices), then the remaining one is matched to one of $\{4, 5, 6, 7, 8, 9\}$(6 choices), and then we choose 4 pairs from the remaining 8 elements(similar to part (b), $7*5*3*1 = 105$ choices). Thus the number of this case is $\binom{3}{2} \cdot 6 \cdot 105 = 1890$. On the contrary, we first match $1, 2, 3$ sequentially to numbers from $\{4, 5, 6, 7, 8, 9\}$($6, 5, 4$ choices each round), then choose 3 pairs from the remaining 6 numbers(by part (b), $5 * 3 * 1 = 15$ choices), thus the number of this case is $6 \cdot 5 \cdot 4 \cdot 15 = 1800$. Therefore, the total number of such pairing is $1890 + 1800 = 3690$.

(f) We first divide even numbers to 3 pairs, by part (b), there are $5 * 3 * 1 = 15$ choices. We then choose 1 from the 3 pairs, and match each even number in this pair to odd numbers, then there are $6, 5$ choices each round. Finally, we divide the remaining 4 odd numbers to 2 groups, and there are $3 * 1$ choices. In total, the number of such pairings should be $15 * 3 * 6 * 5 * 3 = 4050$.

□

Question 4 (Extra credit, 1 point). *Approximately how much time did you spend working on this homework?*