Announcements

- Exam 2 Solutions Online
- No Homework this week
- Lectures this week will be posted on main course webpage

Today

- Exponential Generating Functions

Generating Function that Doesn't Work

Let $a_n = c(n,2)$ (Sterling Number of the first kind).

$$\begin{aligned} a_n &= c(n,2) \\ &= (n-1)c(n-1,2) + c(n-1,1) \\ &= (n-1)a_{n-1} + (n-2)! \end{aligned}$$

Generating function:

$$\begin{aligned} A(x) &= \Sigma_n\, a_n\, x^n = x\, \Sigma_n\, a_{n+1}\, x^n \\ &= \Sigma_n\, n\, a_n\, x^n + \boxed{\Sigma_n\, (n-1)!}\, x^n. \end{aligned}$$

Doesn't Converge!

Exponential Generating Functions

To solve this, we need to rethink our use of generating functions.

Ordinary generating functions:

Store terms in a sequence as coefficients in a power series: $\{a_n\} \leftrightarrow \Sigma_n a_n x^n$.

Why do we need to multiply a_n just by x^n?

Exponential generating functions:

Store terms in a sequence as coefficients in a power series: $\{a_n\} \leftrightarrow \Sigma_n a_n x^n/n!$

Example

$a_{n+1} = na_n + (n-1)!$

$$
\begin{aligned}
A(x) &= \Sigma_n\, a_{n+1}\, x^n/n! \\
&= \Sigma_n\, na_n\, x^n/n! + \Sigma_n\, (n-1)!\, x^n/n! \\
&= \Sigma_n\, a_n\, x^n/(n-1)! + \Sigma_n\, x^n/n \\
&= x\, \Sigma_n\, a_{(n-1)+1}\, x^{n-1}/(n-1)! + \log(1/(1-x)) \\
&= xA(x) + \log(1/(1-x)).
\end{aligned}
$$

So $A(x) = \log(1/(1-x))/(1-x)$.

Translation

What if instead we want the generating function for

$B(x) = \Sigma_n a_n x^n/n!$

$= \Sigma_n a_{n+1} x^{n+1}/(n+1)!$

Notice that $B(x) = \int A(x)dx$.

From this we determine that $B(x) = \log^2(1-x)/2$.

Comparison

Ordinary:

- $F(x) = \Sigma_n a_n x^n$.
- $a_n = 1$, $F(x) = 1/(1-x)$.
- $a_n = n$, $F(x) = x/(1-x)^2$.
- $\Sigma_n a_{n-1} x^n = x F(x)$.
- Converges only if an grows at most exponentially.

Exponential:

- $F(x) = \Sigma_n a_n x^n/n!$
- $a_n = 1$, $F(x) = e^x$.
- $a_n = n$, $F(x) = xe^x$.
- $\Sigma_n a_{n+1} x^n/n! = F'(x)$.
- Converges more generally.

Example

Recurrence: $a_0 = 2$, $a_{n+1} = 2a_n - 1$.

Generating Function: $A(x) = \Sigma_n a_n x^n/n!$

$$
\begin{aligned}
A'(x) &= \Sigma_n a_n x^{n-1}/(n-1)! \\
&= \Sigma_n (2a_{n-1} - 1) x^{n-1}/(n-1)! \\
&= 2\Sigma_n a_{n-1} x^{n-1}/(n-1)! - \Sigma_n x^{n-1}/(n-1)! \\
&= 2A(x) - e^x.
\end{aligned}
$$

Solving: $A(x) = e^x + e^{2x}$.

$A(x) = \Sigma_n x^n/n! + \Sigma_n (2x)^n/n! = \Sigma_n (2^n+1) x^n/n!$

$a_n = 2^n+1$.

Derangements

Lemma: $D_{n+1} = nD_n + nD_{n-1}$.

Proof:

Split into cases:

- n+1 in a cycle of length 2
- n+1 in a cycle of length more than 2.

Case I

If we have a derangement of [n+1] with n+1 in a cycle of length 2:

- n choices of which other element in the cycle.
- D_{n-1} ways to pick a derangement of the other elements.

Total: nD_{n-1}.

Case II

If we have a derangement of [n+1] where n+1 is in a cycle of length more than 2:

Removing n+1 from it's cycle gives a derangement of [n].

- D_n choices for the derangement of [n].
- n ways to insert n+1 into a cycle.

Total: nD_n.

Result

Combining the counts from Case I and Case II we find that

$$D_{n+1} = nD_n + nD_{n-1}.$$

Generating Function

Let $F(x) = \Sigma_n D_n x^n/n!$ be the exponential generating function.

$F'(x) = \Sigma_n D_{n+1} x^n/n!$

$= \Sigma_n nD_n x^n/n! + \Sigma_n nD_{n-1} x^n/n!$

$= xF'(x) + \Sigma_n D_{n-1} x^n/(n-1)!$

$= xF'(x) + xF(x).$

[If $f(x) = \Sigma_n a_n x^n/n!$, $xf'(x) = \Sigma_n na_n x^n/n!$]

Differential Equation

So: $F'(x) = xF'(x) + xF$.

If you remember how to solve differential equations….

$F'(x)/F(x) = x/(1-x) = 1/(1-x) - 1$.

Integrating gives

$\log(F(x)) = - x + \log(1/(1-x)) + \text{const}$.

Since $F(0) = 1$,

$F(x) = e^{-x}/(1-x)$.

Coefficients

$F(x) = e^{-x} / (1-x)$

$= (1 - x + x^2/2! - x^3/3! + \ldots)(1 + x + x^2 + \ldots)$

$= (1+x+x^2+\ldots) - (x+x^2+x^3+\ldots)$

$+1/2(x^2+x^3+x^4+\ldots) -$

$1/6(x^3+x^4+x^5+\ldots)+\ldots$

$= \Sigma_n x^n (1-1+1/2-1/6+\ldots\pm 1/n!)$.

So,

$D_n = n! (1-1+1/2-1/6+\ldots\pm 1/n!)$.

Restricted Set Partitions

Let A_n denote the number of partitions of a set of size n into subsets of size 2.

<u>Recursion:</u>

$A_n = (n-1)A_{n-2}$.

<u>Proof:</u>

The element n pairs with any of n-1 other things. Then need to partition remaining n-2 elements.

Generating Function

Let $G(x) = \Sigma_n A_n x^n/n!$ be the exponential generating function.

$$G'(x) = \Sigma_n A_{n+1} x^n/n!$$
$$= \Sigma_n nA_{n-1} x^n/n!$$
$$= \Sigma_n A_{n-1} x^n/(n-1)!$$
$$= x \Sigma_n A_{n-1} x^{n-1}/(n-1)!$$
$$= xG(x).$$

Differential Equation

$G'(x) = xG(x)$

$G'(x)/G(x) = x$

$\log(G(x)) = x^2/2 + C$

(Noting that $G(0) = A0 = 1$, we have $C = 0$).

So $G(x) = \exp(x^2/2) = 1 + x^2/2 + x^4/(2^2\ 2!) + x^6/(2^3\ 3!) + \ldots$

$x^{2n+1}/(2n+1)!$-coefficient is 0, so $A_{2n+1} = 0$.

$x^{2n}/(2n)!$-coefficient is $(2n)!/(2^n\ n!) = (2n)!/((2n)(2(n-1))(2(n-2))\ldots 2)$.

So $A_{2n} = (2n-1)(2n-3)\ldots 1$.

Multiplication of Exponential Generating Functions

- Multiplication of ordinary generating functions important.
- Multiplication of exponential generating functions a bit different.
- This difference is one of the most important distinguishing features.

Multiplication of Exponential Generating Functions

$A(x) = \Sigma_n\ a_n\ x^n/n!$

$B(x) = \Sigma_n\ b_n\ x^n/n!$

$C(x) = A(x)B(x) = \Sigma_n\ c_n\ x^n/n!$

What is c_n?

Multiplication Formula

$$C(x) = A(x)B(x) = \left(\sum_{m=0}^{\infty} a_m x^m/m!\right)\left(\sum_{k=0}^{\infty} b_k x^k/k!\right)$$

$$= \sum_{m,k=0}^{\infty} a_m b_k x^{m+k}/(m!k!)$$

$$= \sum_{n=0}^{\infty} x^n \left(\sum_{m+k=n} a_m b_k/(m!k!)\right)$$

$$= \sum_{n=0}^{\infty} x^n/n! \left(\sum_{m+k=n} \binom{n}{k} a_m b_k\right).$$

$$c_n = \sum_{k=0}^{n} \boxed{\binom{n}{k}} a_k b_{n-k}.$$

Difference from ordinary formula

Combinatorial Interpretation

Define A-structure a thing so that there are a_n A-structures on a set of size n.

Define B-structure a thing so that there are b_n B-structures on a set of size n.

Ordinary generating function multiplication talks about the number of ways to find an object of A-type and one of B-type of total size n.

Exponential generating function multiplication has c_n = number of ways to partition [n] into two sets and put an A-structure on one and a B-structure on the other.

If A-structure of size k, nCk ways to partition [n], a_k A-structures and b_{n-k} B-structures.

Example: Bell Numbers

In order to get a set partition of [n+1] you need to:

- Partition [n] into:
  - The set of elements that go with n+1
  - The rest
  - Give a set partition of the rest

$$B(n+1) = \sum_{k=0}^{n} \binom{n}{k} B(k).$$

A-structure: Just a set, $a_n = 1$.

B-structure: Set partition $b_n = B(n)$.

Generating Function

Let $F(x) = \Sigma_n B(n)x^n/n!$

$F'(x) = \Sigma_n B(n+1)x^n/n!$

$= \Sigma_n x^n/n! \; \Sigma_k (nCk)B(k)$

$= (\Sigma_m x^m/m!)(\Sigma_k B(k)x^k/k!)$

$= e^x F(x)$

Differential Equation

$F'(x) = e^x F(x)$.

Separating:

$F'(x)/F(x) = e^x$.

Integrating:

$\log(F(x)) = e^x + \text{const}$.

Noting $F(0)=1$, const = -1.

$$\sum_{n=0}^{\infty} B(n)x^n/n! = e^{e^x - 1}.$$

Example II: Proof of Binomial Theorem

Note that: $e^{z(x+y)} = e^{zx} e^{zy}$.

LHS equals:

$\Sigma_n (x+y)^n z^n/n!$

RHS equals:

$[\Sigma_m x^m z^m/m!][\Sigma_k y^k z^k/k!]$

$= \Sigma_n z^n/n! (\Sigma_k (nCk)x^{n-k} y^k)$

Equating the $z^n/n!$ coefficients yields the binomial theorem.

Example III

c_n = c(n,2) = # permutations of [n] with two cycles.

Can think of this permutation as partitioning [n] into two subsets and arranging each subset into a cycle. (actually half of this because we want to unordered cycles)

A-structure is a cycle, as is B-structre.

How many cycles on a set of size n?

(n-1)!

Example III (Continued)

So,

$A(x) = B(x) = \Sigma_{n\geq 1} (n-1)!\, x^n/n! = \Sigma_{n\geq 1} x^n/n = \log(1/(1-x))$.

So the generating function for c_n =
#{ways to partition [n] into two sets and put cycle structure on each}/2
is

$C(x) = \Sigma_n c_n x^n/n! = A(x)B(x)/2 = \log^2(1/(1-x))/2$.

Example IV

Let dn denote the number of ways to partition [n] into sets A, B, C with:

- Elements of A colored red or blue
- |B| even
- C has a selected element

This is the number of ways to partition into subsets each with a specified structure:

- A-structure: color elements red or blue: $a_n = 2^n$.
- B-structure: set must be of even size: $b_n = 0$ if n odd, 1 if n even.
- C-structure: pick single element of set: $c_n = n$.

A(x)

$$A(x) = \Sigma_n a_n x^n/n!$$
$$= \Sigma_n 2^n x^n/n!$$
$$= \Sigma_n (2x)^n/n!$$
$$= e^{2x}.$$

B(x)

$$B(x) = \Sigma_n [1 \text{ if } n \text{ even } 0 \text{ if } n \text{ odd}]\ x^n/n!$$
$$= \Sigma_n (1+(-1)^n)/2\ x^n/n!$$
$$= (1/2)\Sigma_n x^n/n! + (1/2)\ \Sigma_n (-x)^n/n!$$
$$= (1/2)e^x + (1/2)e^{-x}$$
$$= (e^x + e^{-x})/2$$
$$= \cosh(x).$$

$C(x)$

$$\begin{aligned} C(x) &= \Sigma_n\, n\, x^n/n! \\ &= \Sigma_n\, x^n/(n-1)! \\ &= x\, \Sigma_n\, x^{n-1}/(n-1)! \\ &= xe^x \end{aligned}$$

Example IV – Putting it Together

So, d_n is the number of ways to partitions [n] into three subsets, put an A-structure on the first, a B-structure on the second and a C-structure on the third.

$$\Sigma_n d_n x^n/n! = A(x)B(x)C(x)$$
$$= (e^{2x})\cosh(x)(xe^x)$$
$$= x(e^{4x}+e^{2x})/2$$
$$= x\Sigma_n (4^n+2^n)/2\ x^n/n!$$
$$= \Sigma_n n(4^{n-1}+2^{n-1})/2\ x^n/n!$$

$d_n = n(4^{n-1}+2^{n-1})/2.$