Math 184 Homework 6

Spring 2022

This homework is due on gradescope Friday May 27th at 11:59pm pacific time. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in LaTeXis recommend though not required.

Question 1 (Matchings Again, 20 points)**.** *Let* a_n *be the number of matchings on a set of size* n*. Give a way of writing the exponential generating function for* a_n *as a composition. Use this to compute an explicit formula for* a_n*.*

Question 2 (Binary Trees, 20 points)**.** *A* binary tree *is either empty (has no nodes) or has a root node and two more binary trees known as the left and right subtrees. Letting* b_n *be the number of binary trees with nodes labelled* $1, 2, \ldots, n$ *and*

$$B(x) = \sum_{n=0}^{\infty} b_n x^n / n!,$$

show that

$$B(x) = 1 + x(B(x))^2.$$

Conclude that $b_n = n! C_n$*.*

Question 3 (Stack-Sortable Permutations, 30 points)**.** *It is not hard to show directly that the number of stack-sortable permutations of* $[n]$ *is given by the* n^{th} *Catalan number. In particular, if a permutation of* $[n]$ *is stack sortable, there is some pattern of push and pop operations needed to sort it (a push adds the next element in line onto the stack and a pop removes the element on the top of the stack). Find a bijection between such sequences of pushes and pops and lattice paths from* $(0, 0)$ *to* (n, n) *that stay above the line* $x = y$*. Show that each such pattern of pushes and pops corresponds to exactly 1 unique stack-sortable permutation.*

Question 4 (Packing Patterns, 30 points)**.** *Show that for every positive integer* n *there is a permutation* π *of* $[n^2]$ *so that for* every *permutation* ρ *of* $[n]$*, there is a copy of* ρ *inside* π*.*

Question 5 (Extra credit, 1 point)**.** *Free point!*