Math 184 Homework 6

Spring 2022

This homework is due on gradescope Friday May 27th at 11:59pm pacific time. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in \texttt{LATEX} is recommend though not required.

**Question 1** (Matchings Again, 20 points). Let $a_n$ be the number of matchings on a set of size $n$. Give a way of writing the exponential generating function for $a_n$ as a composition. Use this to compute an explicit formula for $a_n$.

**Question 2** (Binary Trees, 20 points). A binary tree is either empty (has no nodes) or has a root node and two more binary trees known as the left and right subtrees. Letting $b_n$ be the number of binary trees with nodes labelled $1, 2, \ldots, n$ and

$$B(x) = \sum_{n=0}^{\infty} b_n x^n / n!,$$

show that

$$B(x) = 1 + x(B(x))^2.$$ 

Conclude that $b_n = n!C_n$.

**Question 3** (Stack-Sortable Permutations, 30 points). It is not hard to show directly that the number of stack-sortable permutations of $[n]$ is given by the $n^{th}$ Catalan number. In particular, if a permutation of $[n]$ is stack sortable, there is some pattern of push and pop operations needed to sort it (a push adds the next element in line onto the stack and a pop removes the element on the top of the stack). Find a bijection between such sequences of pushes and pops and lattice paths from $(0,0)$ to $(n,n)$ that stay above the line $x = y$. Show that each such pattern of pushes and pops corresponds to exactly 1 unique stack-sortable permutation.

**Question 4** (Packing Patterns, 30 points). Show that for every positive integer $n$ there is a permutation $\pi$ of $[n^2]$ so that for every permutation $\rho$ of $[n]$, there is a copy of $\rho$ inside $\pi$.

**Question 5** (Extra credit, 1 point). Free point!