

Math 184 Homework 5

Spring 2022

This homework is due on gradescope Friday May 13th at 11:59pm pacific time. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in L^AT_EX is recommended though not required.

Question 1 (Semi-Increasing Sequences, 20 points). Define a sequence a_1, a_2, \dots, a_n to be semi-increasing if $a_j \geq a_i - 1$ for all $j > i$. Define $\text{sem}(n, k)$ to be the number of semi-increasing sequences of length n consisting of integers from 1 to k . Determine (as a function of k) the generating function

$$F_k(x) = \sum_{n=0}^{\infty} \text{sem}(n, k)x^n.$$

Hint: Proceed by induction on k . Consider the first occurrence (if one exists) of k in the semi-increasing sequence counted by $\text{sem}(n, k)$.

Question 2 (Partition Identity, 20 points). Use generating functions to prove that $p(n) - p(n - 1)$ is the number of partitions of n into parts of size bigger than 1.

Question 3 (Triangulations, 20 points). Define a triangulation of a polygon to be a way of drawing line segments between some pairs of its non-adjacent vertices so that

1. No two of these line segments cross (except possibly at endpoints).
2. These line segments divide the interior of the polygon into triangles.

Prove that the number of triangulations of a convex n -gon is the Catalan number C_{n-2} . *Hint: Show that it satisfies the recurrence relation for the Catalan numbers.*

Question 4 (Colored Compositions, 40 points). .

(a) Let a_n be the number of compositions of n (into any number of parts) in which each part in the composition is colored either red or blue. Give a formula for the generating function

$$\sum_{n=0}^{\infty} a_n x^n.$$

[10 points]

(b) Using the above generating function, obtain a formula for a_n . [10 points]

(c) Let b_n be the number of compositions of n (into any number of parts) in which exactly 2 parts are colored blue. Give a formula for the generating function

$$\sum_{n=0}^{\infty} b_n x^n.$$

[10 points]

(d) Using the above generating function, obtain a formula for b_n . [10 points]

Question 5 (Extra credit, 1 point). Approximately how much time did you spend working on this homework?