Math 184 Homework 4

Spring 2022

This homework is due on gradescope Friday May 6th at 11:59pm pacific time. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in LaTeXis recommend though not required.

Question 1 (Chromatic Polynomials, 40 points)**.** *A* graph *G is a pair of a set V of* vertices*, and a set E of* edges *each connecting two vertices. We call a graph* simple *if no edge connects two of the same vertex and no two edges connect the same pair of vertices. An n-*coloring *of a graph is a way of assigning each vertex a number from $1, 2, \ldots, n$ so that no two vertices connected by an edge are assigned the same number.*

(a) Show that for any finite, simple graph G there is a polynomial $P_G(x)$ so that for any positive integer n, the number of n-colorings of G equals $P_G(n)$. Hint: Use Inclusion-Exclusion. [20 points]

(b) What are the three highest degree terms of $P_G(x)$ in terms of properties of the graph G? [20 points]

Question 2 (Reverse Inclusion-Exclusion, 20 points)**.** *For finite sets A, B, and C give a formula for $|A \cap B \cap C|$ in terms of $|A|, |B|, |C|, |A \cup B|, |B \cup C|, |C \cup A|, |A \cup B \cup C|$.*

Question 3 (Finite Differences of $1/x$, 10 points)**.** *For n a positive integer, give a formula for*

$$\sum_{k=0}^{n} \frac{(-1)^k}{k+1} \binom{n}{k}.$$

Hint: Integrate the binomial theorem.

Question 4 (Linear Homogeneous Recurrence Relations, 30 points)**.** *We say that a sequence of numbers $a_0, a_1, a_2, \ldots$ satisfies a linear homogeneous recurrence relation with constant coefficients if there exists a positive integer k and real numbers $c_1, c_2, \ldots, c_k$ so that for all sufficiently large integers n*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}.$$

Show that a sequence of real numbers $a_0, a_1, a_2, \ldots$ satisfies a linear homogeneous recurrence relation with constant coefficients if and only if the corresponding generating function

$$A(x) := \sum_{n=0}^{\infty} a_n x^n$$

is a rational function (i.e. is the ratio of two polynomials in x).

Question 5 (Extra credit, 1 point)**.** *Approximately how much time did you spend working on this homework?*