Math 184 Homework 4

Spring 2022

This homework is due on gradescope Friday May 6th at 11:59pm pacific time. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in \LaTeX is recommended though not required.

Question 1 (Chromatic Polynomials, 40 points). A graph $G$ is a pair of a set $V$ of vertices, and a set $E$ of edges each connecting two vertices. We call a graph simple if no edge connects two of the same vertex and no two edges connect the same pair of vertices. An $n$-coloring of a graph is a way of assigning each vertex a number from $1, 2, \ldots, n$ so that no two vertices connected by an edge are assigned the same number.

(a) Show that for any finite, simple graph $G$ there is a polynomial $P_G(x)$ so that for any positive integer $n$, the number of $n$-colorings of $G$ equals $P_G(n)$. Hint: Use Inclusion-Exclusion. [20 points]

(b) What are the three highest degree terms of $P_G(x)$ in terms of properties of the graph $G$? [20 points]

Question 2 (Reverse Inclusion-Exclusion, 20 points). For finite sets $A, B$, and $C$ give a formula for $|A \cap B \cap C|$ in terms of $|A|, |B|, |C|, |A \cup B|, |B \cup C|, |C \cup A|, |A \cup B \cup C|$.

Question 3 (Finite Differences of $1/x$, 10 points). For $n$ a positive integer, give a formula for

$$
\sum_{k=0}^{n} \frac{(-1)^k}{k+1} \binom{n}{k}.
$$

Hint: Integrate the binomial theorem.

Question 4 (Linear Homogeneous Recurrence Relations, 30 points). We say that a sequence of numbers $a_0, a_1, a_2, \ldots$ satisfies a linear homogeneous recurrence relation with constant coefficients if there exists a positive integer $k$ and real numbers $c_1, c_2, \ldots, c_k$ so that for all sufficiently large integers $n$

$$
a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_k a_{n-k}.
$$

Show that a sequence of real numbers $a_0, a_1, a_2, \ldots$ satisfies a linear homogeneous recurrence relation with constant coefficients if and only if the corresponding generating function

$$
A(x) := \sum_{n=0}^{\infty} a_n x^n
$$

is a rational function (i.e. is the ratio of two polynomials in $x$).

Question 5 (Extra credit, 1 point). Approximately how much time did you spend working on this homework?