Math 184 Homework 3

Spring 2022

This homework is due on gradescope Friday April 22nd at 11:59pm pacific time. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in LaTeXis recommend though not required.

Question 1 (Matchings and Permutations, 20 points)**.** .

(a) *How many permutations of* $[2n]$ *consist of* n *cycles of length 2? [5 points]*

(b) *Show a bijection between such permutations and matchings of* $[2n]$*. [5 points]*

(c) *How many permutations of* $[3n]$ *consist of* n *cycles of length 3? [5 points]*

(d) *Does the answer in part c equal to the number of partitions of* $[3n]$ *into sets of size* 3*? Why or why not? [5 points]*

Question 2 (Square Permutations, 20 points)**.** *Let* π *be a permutation of* $[n]$*. Show that there exists a permutation* σ *with* $\pi = \sigma^2$ *if and only if* π *has a even number of cycles of length* k *for every even number* k*.*

Question 3 (Largest in its Cycle, 30 points)**.** *For integers* $1 \leq k \leq n$*, how many permutations of* $[n]$ *have* k *as the largest element in its cycle? Hint: Consider the canonical cycle representation of such permutations.*

Question 4 (Sterling Number Bounds, 30 points)**.** *Show the following size bounds on Sterling numbers of the first kind for* $1 \leq k \leq n$*:*

(a) $c(n,k) \geq (n-1)!/(k-1)!$ *[10 points]*

(b) *For any positive integer* a*,*

$$c(n,k) \leq n!\binom{n+a}{a}/(a+1)^k.$$

Hint: you may want to use the relation that

$$\sum_{k=1}^{n} x^k c(n,k) = x(x+1)\cdots(x+n-1).$$

[20 points]

Question 5 (Extra credit, 1 point)**.** *Approximately how much time did you spend working on this homework?*