Math 184 Homework 1

Spring 2022

This homework is due on gradescope Friday April 8th at 11:59pm pacific time. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in LaTeXis recommend though not required.

Question 1 (Symmetric Polynomials, 50 points). *Call a polynomial P in the variables $x_1, x_2, \dots, x_n$* symmetric *if switching any of the variables leaves P unchanged. So for example $x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3$ is a symmetric polynomial in x_1, x_2, x_3 but $x_1 + 2x_2 + 3x_3$ is not. A particular example of this are the* power-sum symmetric polynomials *defined as $p_k = \sum_{i=1}^n x_i^k$. Show that any symmetric polynomial can be written as a polynomial in the power-sum symmetric polynomials. For example, if $P(x, y, z) = x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3$, then $P = p_2 - p_1^3/6 + p_1p_2/2 - p_3/3$.*

Hint: You will want to use induction, but not on the number of variables. Start with a polynomial P and find a way to add or subtract products of the power-sum polynomials to simplify it. Repeat this until there is nothing left.

Question 2 (Simultaneous Rational Approximation, 20 points). *Dirichlet's Theorem is useful when you want to approximate one number by rationals, but what if you have two? Suppose that you have two real numbers x and y and want to find integers n, k, m so that $|x - n/m|$ and $|y - k/m|$ are both small. Prove that for any integer q, one can always find n, k, m with $|m| \leq q^2$ so that $|x - n/m|$ and $|y - k/m|$ are each at most $1/(mq)$.*

Question 3 (Counting Matchings, 30 points). *Let $[12]$ denote the set $\{1, 2, 3, \dots, 12\}$. A* matching *of $[12]$ is a way of partitioning the elements into pairs so that each element is in exactly one pair. For example, one matching is $\{1, 3\}, \{2, 7\}, \{4, 10\}, \{5, 6\}, \{8, 11\}, \{9, 12\}$. For each of the following count the number of matchings with the given property both as a formula and by giving the exact number. Remember to justify your answer.*

(a) The number of all matchings of $[12]$. [5 points]

(b) The number of matchings of $[12]$ where each even number is paired with another even number. [5 points]

(c) The number of matchings of $[12]$ where each even number is paired with an odd number. [5 points]

(d) The number of matchings of $[12]$ where each number is paired with another number at most 2 away from it (for this you will want to relate the number of such pairings of $[2n]$ to the number of such pairings of $[2(n-1)]$ and of $[2(n-2)]$ and produce a recurrence). [5 points]

(e) The number of matchings of $[12]$ where each of $1, 2, 3$ is paired to one of $1, 2, 3, 4, 5, 6, 7, 8, 9$. [5 points]

(f) The number of matchings of $[12]$ where there are exactly 2 pairs of even numbers that are matched together. [5 points]

Question 4 (Extra credit, 1 point). *Approximately how much time did you spend working on this homework?*