Question 1 (Canonical Cycle Notation, 25 points)**.** *A permutation of* [9] *when written in function notation is* 879231465*. What is this permutation when written in canonical cycle notation? Express your answer as a single* 9*-digit number without parentheses.*

The answer is 742861953.

The permutation produces cycles $7 \to 4 \to 2$, $8 \to 6 \to 1$ and $9 \to 5 \to 3$. Writing each permutation starting with its largest elements and writing them in increasing order of this largest element gives (742)(861)(953). Removing the parentheses yields the answer.

Question 2 (Coefficient Calculation, 25 points)**.** *What is the coefficient of xy^3z^5 in $(x+y+z)^9$?*

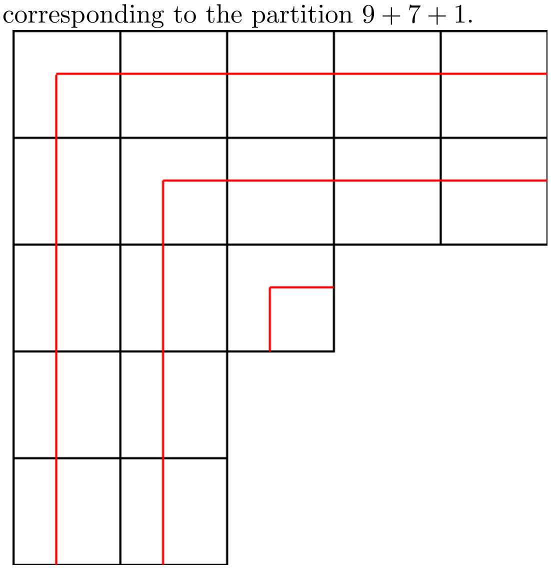
The answer is 504.
The Multinomial Theorem says that the answer is

$$\binom{9}{1,3,5} = \frac{9!}{5!3!1!} = 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 15\cdot 4\cdot 3\cdot 2\cdot 1\cdot 3\cdot 2\cdot 1 = \frac{9\cdot 8\cdot 7\cdot 6}{6} = 9\cdot 8\cdot 7 = 504.$$

Question 3 (Partition Correspondence, 25 points)**.** *The partition* $17 = 5 + 5 + 3 + 2 + 2$ *is self-conjugate. What is the corresponding partition of* 17 *into distinct, odd parts? Express your answer as a number obtained by listing the parts in descending order (note that each part should be a 1-digit number). For example, the partition* $8 + 7 + 6$ *would correspond to the answer* 876*.*

The answer is 971.

The Ferrers diagram for the partition is drawn below. Dividing it into L's we find ones of length $9, 7$ and 1, corresponding to the partition $9 + 7 + 1$.



Question 4 (Promotion Fatigue, 25 points)**.** *A particular store runs a number n of different promotions in February, each lasting a single day. How big does n need to be to guarantee that every year at least three of these promotions will take place during the same calendar week (a calendar week runs from Sunday until the next Saturday)?*

The answer is 11.

February always contains days from at most 5 different calendar weeks. Thus, the generalized pigeonhole principle says that if we have at least $(3-1)5+1=11$ promotions that at least three must lie in the same week. It is not hard to produce examples where 10 promotions are not enough.

Question 5 (Set Union Size Range, 25 points)**.** *Suppose that* $A, B,$ *and* C *are three sets with* $|A| = 100$, $|B| = 75$, $|C| = 80$ *and* $|A \cap B| = 55$, $|B \cap C| = 30$. *What is the smallest possible size of* $A \cup B \cup C$*?*

The answer is 125.

We note that $|C \backslash B| = |C| - |C \cap B| = 50$ and that $|A \backslash B| = |A| - |A \cap B| = 45$. It is entirely possible that the points in $A \backslash B$ are entirely contained in the points of $C \backslash B$. If this is the case, then $A \cup B \cup C = B \cup C$, which has size $|B \cup C| = |B| + |C| - |B \cap C| = 75 + 80 - 30 = 123$.

Question 6 (Cycle Notation Count, 25 points). *On average, in how many ways can a permutation π of* [9] *be written in (not necessarily canonical) cycle notation?*

The answer is 256.

The total number of permutation of [9] is 9!. If C is the number of ways to write one of these permutations in a cycle notation, the answer will be $C/9!$. In order to write a permutation in cycle notation, we can first put the numbers $1, 2, \ldots, 9$ in order in one of 9! ways, and then we need to group them into cycles. Such a grouping into cycles corresponds to a composition of 9. There are $2^{9-1} = 256$ such compositions. Thus the answer is $256 \cdot 9!/9! = 256$.

Question 7 (Set Partition Counting, 25 points)**.** *Let f_n be the number of ways to partition a set $[n]$ into subsets A, B, C so that:*

- *The elements of A are colored red, blue and green*
- *B is non-empty*
- *$|C|$ is even.*

Let

$$F(x) = \sum_{n=0}^{\infty} f_n x^n/n!$$

be the corresponding exponential generating function. What is $F(\log(3))$, where $\log(3)$ denotes the natural logarithm of 3?

The answer is 90.

f_n counts the number of ways to partition a set of size n into three subsets, put an A-structure on the first, a B-structure on the second and a C-structure on the third where an A-structure is coloring the elements red, green and blue, a B-structure is a non-empty set, and a C-structure is a set of even size. Thus, $F(x)$ is the product $A(x)B(x)C(x)$ where $A(x)$, $B(x)$ and $C(x)$ are the exponential generating functions corresponding to the $A-, B-$, and $C-$structures.

There are 3^n $A-$structures on a set of size n so

$$A(x) = \sum_{n=0}^{\infty} 3^n x^n/n! = e^{3x}.$$

There is one B-structure on a set of size n for each $n \geq 1$ and none for $n = 0$, thus,

$$B(x) = \sum_{n=1}^{\infty} x^n/n! = e^x - 1.$$

There is one C-structure on a set of size n if n is even and none otherwise, thus

$$C(x) = \sum_{n=0,\text{ even}}^{\infty} x^n/n! = \cosh(x).$$

Thus,

$$F(x) = e^{3x}(e^x - 1)\cosh(x) = e^{3x}(e^x - 1)(e^x + e^{-x})/2 = (e^{5x} - e^{4x} + e^{3x} - e^{2x})/2.$$

Plugging in $x = \log(3)$, we get

$$F(\log(3)) = (3^5 - 3^4 + 3^3 - 3^2)/2 = 90.$$

Question 8 (One Even Part, 25 points)**.** *Let g_n be the number of weak compositions of n where exactly one part of the composition is even. For example, $g_2 = 4$ due to the compositions $2, 1+1+0, 1+0+1, 0+1+1$. Let $G(x)$ be the generating function*

$$G(x) = \sum_{n=0}^{\infty} g_n x^n.$$

What is $G(1/2)$?

The answer is 12.

Let a_n be the number of weak compositions of n into odd parts and one zero part. One can construct this by first choosing a composition of n into some k odd parts and then choosing a location for the zero part. Note that this last step can be done in $k+1$ ways. This makes the generating function $A(x) = \sum_{n=0}^{\infty} a_n x^n$ a composition of the generating functions $B(x) = \sum_{k=0}^{\infty}(k+1)x^k = 1/(1-x)^2$ and $C(x) = \sum_{n \text{ odd}} x^n = x/(1-x^2)$.

To get a composition of the part we want, we need to add to a composition of this form an even number representing the size of the even part. Thus $G(x)$ is a product of generating functions $G(x) = A(x)D(x)$ where $D(x) = \sum_{n \text{ even}} x^n = 1/(1-x^2)$. Thus,

$$G(x) = 1/(1-x^2)\left(\frac{1}{(1-x/(1-x^2))^2}\right) = \frac{1-x^2}{(1-x-x^2)^2}.$$

Evaluating at $x = 1/2$ gives

$$G(1/2) = (3/4)/(1/4)^2 = 12.$$