Math 184

Exam 3 Review

Fall 2026

Exam Format

- In class
- 4 Qs in 50 minutes
- 6 one-sided pages of notes
- No other written or electronic aids (other than a simple timepiece)
- Randomized assigned seating

Review Materials

- This presentation (a brief overview of topics)
- Lecture videos
- Office hours
- Textbook
- Problem archive

Binomial Theorem and Related Identities (Chapter 4)

- Binomial Theorem
- Applications
- Generalizations

The Binomial Theorem

Theorem: For integers n ≥ 0, and real numbers x and y, we have that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof

We can prove the binomial theorem by:

- Induction on n
- Expanding out $(x+y)^n$.
- Combinatorial Proof

Application 1

Start with the binomial theorem:

$(x+y)^n = \Sigma_k \, (nCk) x^k y^{n-k}$.

Differentiate with respect to x:

$n(x+y)^{n-1} = \Sigma_k \, k(nCk) x^{k-1} y^{n-k}$.

Substitute $x = y = 1$.

$n2^{n-1} = \Sigma_k \, k \, (nCk)$.

Application 2

Note that $(x+y)^{n+m} = (x+y)^n (x+y)^m$.

What does the binomial theorem say about this?

$$\begin{aligned}\sum_{k=0}^{n+m} \binom{n+m}{k} x^k y^{n+m-k} &= \left(\sum_{a=0}^{n} \binom{n}{a} x^a y^{n-a}\right)\left(\sum_{b=0}^{m} \binom{n}{b} x^b y^{m-b}\right) \\ &= \sum_{a,b} \binom{n}{a}\binom{m}{b} x^{a+b} y^{n+m-a-b} \\ &= \sum_{k} \left(\sum_{a+b=k} \binom{n}{a}\binom{m}{b}\right) x^k y^{n+m-k}.\end{aligned}$$

Comparing terms:

$$\binom{n+m}{k} = \sum_{a+b=k} \binom{n}{a}\binom{m}{b}.$$

Ratios of Binomial Coefficients

Recall (nCk) = n!/(k! (n-k)!).

$$\begin{aligned}
\binom{n}{k} \Big/ \binom{n}{k+1} &= \left(\frac{n!}{k!(n-k)!} \right) / \left(\frac{n!}{(k+1)!(n-k-1)!} \right) \\
&= \left(\frac{(k+1)!}{k!} \right) / \left(\frac{(n-k)!}{(n-k-1)!} \right) \\
&= \frac{k+1}{n-k}.
\end{aligned}$$

Relative Sizes of Binomial Coefficients

$(nCk) / (nC(k+1)) = (k+1) / (n-k)$.

Thus, $(nCk) > (nC(k+1))$ if and only if
$(k+1) > (n-k)$.
This happens if and only if $k \geq n/2$.

Thus, binomial coefficients (nCk) increase in k for $k<n/2$ and then decrease in k for $k>n/2$.

Multinomial Theorem

What is $(x_1+x_2+\ldots+x_k)^n$?

- Expand out, get a sum of terms of the form $x_1^{a1}x_2^{a2}\ldots x_k^{ak}$.
  - Must have $a_1+a_2+\ldots+a_k = n$.
- What is the coefficient of this term?
  - Number of ways to pick x_1 coefficient from a_1 terms, pick x_2 from a_2 terms, etc.
  - Given by multinomial coefficient

$$\binom{n}{a_1, a_2, \ldots, a_k} = \frac{n!}{a_1!a_2!\ldots a_k!}.$$

Non-Integer n

Binomial Theorem with y = 1.

$(1+x)^n = 1 + nx + n(n-1)x^2/2 + \ldots + (n)_k\, x^k/k! + \ldots$

When n is a positive integer this gives a finite sum that we have proved correct.

What happens when n is not an integer?

$(1+x)^a = 1 + ax + a(a-1)x^2/2 + \ldots + (a)_k\, x^k/k! + \ldots$

$$\sqrt{1-4x} = 1 - 2x - 2x^2\binom{2}{1}/2 - 2x^3\binom{4}{2}/3 - 2x^4\binom{6}{3}/4 - \ldots$$

Generating Functions (Chapter 8)

- Basic Idea
- Ordinary Generating Functions
  - Recurrence Relations
  - Binomial Coefficients Identities
  - Catalan Numbers
  - Partition Numbers
- Exponential Generating Functions
  - Multiplication Formula
  - “Hands” Method
  - Sterling Numbers

Generating Functions

Basic Method: In order to study some interesting sequence of numbers, $a_0, a_1, a_2, \ldots$ instead turn these numbers into a single function: $f(x) = a_0 + a_1x + a_2x^2 + \ldots$ and study $f(x)$.

This function $f(x)$ is called a generating function for the sequence $\{a_i\}$.

Generating Functions

Why is this a good idea?

1) The function f can be encode the entire sequence $\{a_i\}$ with a single function.
2) Complicated combinatorial properties of the sequence $\{a_i\}$ can often by encoded as algebraic or analytic properties of f(x).
3) This often lets us reduce solving complicated problems to basic algebra.

Application 1: Recurrence Relations

A recurrence relation is a way of defining a sequence of numbers by giving a formula for each element in terms of previous ones.

Basic Tools

- If two generating functions are the same, the coefficients are the same:
  If $\Sigma_n a_n x^n = \Sigma_n b_n x^n$ then $a_n = b_n$ for all n.
- **Geometric series:**
  $1/(1-cx) = \Sigma_n (cx)^n = \Sigma_n c^n x^n$.
- **Sums of Generating Functions:**
  $(\Sigma_n a_n x^n) + (\Sigma_n b_n x^n) = \Sigma_n (a_n + b_n) x^n$
- **Shifts:** If $F(x) = \Sigma_n a_n x^n$,
  $xF(x) = \Sigma_n a_n x^{n+1} = \Sigma_{n>0} a_{n-1} x^n$.

Partial Fractions

Given a generating function $F(x) = p(x)/q(x)$ with $p(x)$ and $q(x)$ polynomials, how do we find a formula for the coefficients?

1) **<u>Reduce p:</u>** Write $F(x) = r(x) + p_0(x)/q(x)$ with $\text{degree}(p_0) < \text{degree}(q)$.
2) **<u>Factor q:</u>** Write $q(x) = (x-r_1)(x-r_2)\ldots(x-r_k)$.
3) **<u>Rewrite:</u>** Write $p_0(x)/q(x)$ as
   $A_1/(x-r_1) + A_2/(x-r_2) + \ldots + A_k/(x-r_k)$
   [You can always do this if q has distinct roots, otherwise, it is a bit more complicated].

Partial Fractions

4) <u>Get Formula:</u>

Note that:

$1/(x-r_i) = (-1/r_i) / (1-x/r_i)$

$= -(1/r_i) - (1/r_i)^2 x - (1/r_i)^3 x^2 + \ldots$

Adding terms together

$A_1/(x-r_1) + A_2/(x-r_2) + \ldots + A_k/(x-r_k) =$

$\Sigma_n - (A_1/r_1^{n+1} + A_2/r_2^{n+1} + \ldots + A_k/r_k^{n+1})x^n$

To Solve Simple Recurrence Relations

1) Define generating function F(x).
2) Use recurrence relation to relate F(x) to shifted generating functions.
3) Solve for F(x) as a rational function.
4) Compute a partial fraction decomposition.
5) Use to obtain a formula for coefficients.

Products of Generating Functions

Let A and B be sets of non-negative integers and let

$f_A(x) = \Sigma_{a\in A}\, x^a, \quad f_B(x) = \Sigma_{b\in B}\, x^b.$

Then,

$f_A(x)\cdot f_B(x) = \Sigma\, x^{a+b}$

$= \Sigma_n\, x^n\, \#\{a\in A, b\in B: a+b = n\}.$

Coefficients of the product are the number of ways to write n as a sum of elements.

Integer Partitions

An integer partition of n is a way of writing n as the sum of some number of 1s plus some number of 2s and so on.

In other words, it's the number of ways to write n as $a_1 + 2a_2 + 3a_3 + \ldots$ for non-negative integers a_k.

Note that: $1/(1-x^k) = \Sigma_a\, x^{ka}$.

So, $1/(1-x)(1-x^2)(1-x^3)\ldots = \Sigma_n\, p(n)x^n$.

General Products of Generating Functions

$A(x) = \Sigma_n a_n x^n$,

$B(x) = \Sigma_n b_n x^n$.

$A(x) \cdot B(x) = C(x) = \Sigma_n c_n x^n$.

What is c_n?

$$C(x) = (\Sigma_n a_n x^n)(\Sigma_m b_m x^m)$$
$$= \Sigma_{n,m} a_n b_m x^{n+m}$$
$$= \Sigma_k x^k (\Sigma_{n+m=k} a_n b_m)$$

$c_k = (\Sigma_{n+m=k} a_n b_m)$.

Combinatorial Interpretation

Suppose that you have objects of type-A and objects of type-B. Each has a size which is a non-negative integer, and there are a_n objects of type-A of size n, and b_m objects of type-B of size m.

Then c_k is the number of ways to find a pair of an object of type-A and an object of type-B where the sum of the sizes is k.

Catalan Numbers

How many such paths are there?

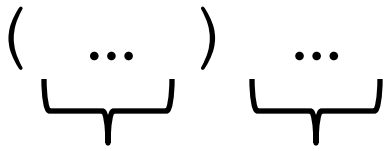
Definition: The nth Catalan Number C_n is the number of up-left lattice paths from (0,0) to (n,n) that stay on or above the line x = y.

Note: Catalan numbers count many other things including matching parentheses sequences.

Recursion

To get started on counting, which parentheses does the opening ‘ (‘ match with?

Matches with the k^{th} ‘) ’ for some k.



- C_{k-1} ways to match first group.
- C_{n-k} ways to match second group.

$$C_n = \sum_{k=1}^{n} C_{k-1} C_{n-k}.$$

Generating Function

Define the generating function:

$H(x) = \Sigma_n C_n x^n$.

$H^2(x) = \Sigma_n (\Sigma_k C_k C_{n-k}) x^n$

$= \Sigma_n C_{n+1} x^n$.

$xH^2(x) = \Sigma_n C_{n+1} x^{n+1} = H(x) - 1$.

Quadratic Formula

We have

$xH^2(x) - H(x) + 1 = 0.$

What is H(x)?

$$H(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x}.$$

Note: Only the – term makes sense, since H(0) needs to be finite.

Coefficients

Recall:

$$\sqrt{1-4x} = 1 - 2x - 2x^2\binom{2}{1}/2 - 2x^3\binom{4}{2}/3 - 2x^4\binom{6}{3}/4 - \dots$$

So,

$$\frac{1-\sqrt{1-4x}}{2x} = 1 + x\binom{2}{1}/2 + x^2\binom{4}{2}/3 + x^3\binom{6}{3}/4 + \dots$$

Therefore,

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Composition of Generating Functions

Suppose:

- $F(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$
- $G(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \ldots$
- $F(G(x)) = c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots$

What is the combinatorial interpretation of the c_i's?

Note: We almost always want to take $b_0 = 0$ for this. Otherwise, $F(G(x))$ isn't necessarily defined even at $x = 0$.

Composition of Generating Functions

$F(G(x)) = a_0 + a_1 G(x) + a_2 G^2(x) + a_3 G^3(x) + \ldots$

Let $G^k(x)$ have coefficients given by

$G^k(x) = \Sigma_n \, c_{n,k} \, x^n$.

We then have that

$c_n = \Sigma_k \, a_k \, c_{n,k}$.

What is the combinatorial interpretation of the $c_{n,k}$?

Powers to Compositions

So we have an interpretation of $c_{n,k}$. What about $c_n = \Sigma_k a_k c_{n,k}$?

Define an A-structure on a set to be a thing where there are a_k ways to build an A-structure on a set of size k.

Then we have that c_n is the number of ways to find an ordered list of B-structures of total size n and then build an A-structure on top of them.

Exponential Generating Functions

To solve this, we need to rethink our use of generating functions.

Ordinary generating functions:

Store terms in a sequence as coefficients in a power series: $\{a_n\} \longleftrightarrow \Sigma_n\, a_n\, x^n$.

Why do we need to multiply a_n just by x^n?

Exponential generating functions:

Store terms in a sequence as coefficients in a power series: $\{a_n\} \longleftrightarrow \Sigma_n\, a_n\, x^n/n!$

Comparison

Ordinary:

- $F(x) = \Sigma_n a_n x^n$.
- $a_n = 1$, $F(x) = 1/(1-x)$.
- $a_n = n$, $F(x) = x/(1-x)^2$.
- $\Sigma_n a_{n-1} x^n = x F(x)$.
- Converges only if an grows at most exponentially.

Exponential:

- $F(x) = \Sigma_n a_n x^n/n!$
- $a_n = 1$, $F(x) = e^x$.
- $a_n = n$, $F(x) = xe^x$.
- $\Sigma_n a_{n+1} x^n/n! = F'(x)$.
- Converges more generally.

Multiplication of Exponential Generating Functions

- Multiplication of ordinary generating functions important.
- Multiplication of exponential generating functions a bit different.
- This difference is one of the most important distinguishing features.

Multiplication Formula

$$C(x) = A(x)B(x) = \left(\sum_{m=0}^{\infty} a_m x^m/m!\right)\left(\sum_{k=0}^{\infty} b_k x^k/k!\right)$$

$$= \sum_{m,k=0}^{\infty} a_m b_k x^{m+k}/(m!k!)$$

$$= \sum_{n=0}^{\infty} x^n \left(\sum_{m+k=n} a_m b_k/(m!k!)\right)$$

$$= \sum_{n=0}^{\infty} x^n/n! \left(\sum_{m+k=n} \binom{n}{k} a_m b_k\right).$$

$$c_n = \sum_{k=0}^{n} \boxed{\binom{n}{k}} a_k b_{n-k}.$$

Difference from ordinary formula

Combinatorial Interpretation

Define A-structure a thing so that there are a_n A-structures on a set of size n.

Define B-structure a thing so that there are b_n B-structures on a set of size n.

Ordinary generating function multiplication talks about the number of ways to find an A-structure and a B-structure of total size n.

Exponential generating function multiplication has c_n = number of ways to partition [n] into two sets and put an A-structure on one and a B-structure on the other.

If A-structure of size k, nCk ways to partition [n], a_k A-structures and b_{n-k} B-structures.

Composition of Generating Functions

So if $A(x) = a_1x + a_2x^2/2! + a_3x^3/3! + \ldots$ and $B(x) = b_0 + b_1x + b_2x^2/2! + b_3x^3/3! + \ldots$

What is $B(A(x))$?

It equals $b_0 + b_1A(x) + b_2A(x)^2/2! + b_3A(x)^3/3! + \ldots$

The $x^n/n!$ coefficient is:

b1 times the number of partitions of [n] into one part with an A-structure plus

b2 times the number of partitions of [n] into two parts with an A-structure on each plus

b3 times the number of partitions of [n] into three parts with an A-structure on each plus …

Composition of Generating Functions

So the $x^n/n!$ coefficient of B(A(x)) counts the number of ways to partition [n] into subsets, put an A-structure on each subset, and put a B-structure on the collection of subsets.