Question 1 (Computing Coefficients, 25 points)**.** *What are the first four coefficients (the x^0 coefficient through the x^3 coefficient) of the generating function*

$$(1+x)^{1/3}/(1-x/3)?$$

We have that

$$(1+x)^{1/3} = 1 + (1/3)x - (1/9)x^2 + (5/81)x^3 + \dots$$

We have that

$$1/(1-x/3) = 1 + (1/3)x + (1/9)x^2 + (1/27)x^3 + \dots$$

Taking the product we have that

$$(1+x)^{1/3}/(1-x/3) = 1 + (2/3)x + (1/9)x^2 + (8/81)x^3 + \dots$$

Question 2 (Recurrence Relation, 25 points)**.** *Suppose that* a_n *is a sequence of integers so that* $a_{n+2} = a_{n+1} - na_n + 1$ *for all* $n \geq 0$*. Letting*

$$A(x) = \sum_{n=0}^{\infty} a_n x^n / n!$$

give a differential equation satisfied by $A(x)$*.*

Differentiating, we find that

$$A''(x) = \sum_{n=0}^{\infty} a_{n+2} x^n / n!$$

Also

$$A'(x) = \sum_{n=0}^{\infty} a_{n+1} x^n / n!$$

Finally

$$xA'(x) = \sum_{n=0}^{\infty} a_{n+1} x^{n+1} / n! = \sum_{n=0}^{\infty} n a_n x^n / n!$$

Noting that $e^x = \sum x^n / n!$, the recurrence tells us that

$$A''(x) = A'(x) - xA'(x) + e^x.$$

Question 3 (Generating Function Product, 25 points)**.** *Let c_n be the number of ways of coloring the numbers $1, 2, \ldots, n$ each red, green or blue so that the number of red numbers equals the number of blue numbers and so that the number of green numbers is odd. Show how to write the exponential generating function for c_n as a product of generating functions whose coefficients can be written down explicitly.*

We note that c_n is the number of ways to partition $[n]$ into two sets S_1 and S_2 and paint half the elements in S_1 red and half blue, and paint the elements of S_2 (of which there must be an odd number) green. The number of ways to color S_1 is $\binom{|S_1|}{|S_1|/2}$ if $|S_1|$ is even (and 0 otherwise) and the number of ways to color S_2 is 1 if $|S_2|$ is odd and 0 otherwise. Thus, we have that

$$\sum_{n=0}^{\infty} c_n x^n/n! = \left(\sum_{n \text{ even}} \binom{n}{n/2} x^n/n!\right)\left(\sum_{n \text{ odd}} x^n/n!\right) = \left(\sum_{n=0}^{\infty} x^{2n}/(n!)^2\right)\left(\sum_{n=0}^{\infty} x^{2n+1}/(2n+1)!\right).$$

Question 4 (Catalan Number Parity, 25 points)**.** *Prove that the Catalan number C_n is odd if and only if n is one less than a power of 2.*

Hint: Use the recurrence relation.

We prove by strong induction on n that C_n is odd if and only if $n+1$ is a power of 2.

We being with the case of $n = 0$. Here $n + 1 = 1$ is a power of 2 and $C_n = 1$ is odd.

For the inductive step, we assume that our statement holds for all smaller values of n. We note that

$$C_n = C_0C_{n-1} + C_1C_{n_2} + \ldots + C_{n-1}C_0.$$

If n is even, this is

$$2(C_0C_{n-1} + C_1C_{n-2} + \ldots + C_{n/2-1}C_{n/2})$$

which is even. Since $n + 1$ is not a power of 2 we are done.

If n is odd, this is

$$2(C_0C_{n-1} + C_1C_{n-2} + \ldots + C_{(n-3)/2}C_{(n+1)/2}) + C_{(n-1)/2}^2.$$

This is odd if and only if $C_{(n-1)/2}$ is. By the inductive hypothesis, this happens if and only if $(n + 1)/2$ is a power of 2, which happens if and only if $n + 1$ is a power of 2. This completes our proof.