Question 1 (Power Series, 25 points). What are the first five terms (i.e. the $x^0$ through $x^4$ terms) of the power series for $(1 - 2x)^{3/2}$ about $x = 0$?

Using Newton’s Formula we have that

$$(1 - 2x)^{3/2}$$

$= 1 + (3/2)(-2x) + (3/2)(1/2)(-2x)^2/2 + (3/2)(1/2)(-1/2)(-2x)^3/6 + (3/2)(1/2)(-1/2)(-3/2)(-2x)^4/24 + \ldots$

$= 1 - 3x + (3/2)x^2 + (1/2)x^3 + (3/8)x^4 + \ldots$
Question 2 (Subset Counting, 25 points). How many subsets of \( \{1,2,\ldots,7\} \) either:

- Contain 1, 2, and 3,
- Contain 3, 4, and 5, OR
- Contain exactly five elements?

Let \( A \) be the set of subsets containing 1,2,3, \( B \) the set of subsets containing 3,4,5, and \( C \) the set of subsets with 5 elements. By Inclusion-Exclusion, we have that

\[
|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|.
\]

We have \( |A| = 2^4 = 16, \ |B| = 2^4 = 16, \ |C| = \binom{7}{3} = 21, \ |A \cap B| = 2^2 = 4, \ |B \cap C| = \binom{4}{2} = 6, \ |C \cap A| = \binom{4}{2} = 6, \ |A \cap B \cap C| = 1. \) Therefore the answer is

\[
16 + 16 + 21 - 4 - 6 - 6 + 1 = 38.
\]
Question 3 (Same Cycle Counting, 25 points). Let $i, j, k$ be three distinct numbers between 1 and $n$. For how many permutations of $[n]$ is $k$ in the same cycle as $i$ but not the same cycle as $j$. Justify your answer.

We note that we get the same answer no matter what $i, j, k$ are, so we may assume that $k = n$. Writing the permutation in canonical cycle notation, we note that $k$ is in the same cycle as $i$ but not the same cycle as $j$ if and only if $j$ comes before $k$ comes before $i$ in canonical cycle notation. To construct such a canonical cycle notation, there are $\binom{n}{3}$ ways to select the three locations for $i, j, k$ (and then $j$ goes on the first of these, $k$ in the second and $i$ in the third), and $(n-3)!$ ways to arrange the other elements. Thus the answer is $\binom{n}{3}(n-3)! = n!/6$. 


Question 4 (Partitions with Distinct Part Sizes, 25 points). Let \( p(n, k) \) denote the number of integer partitions of \( n \) with exactly \( k \) parts of different sizes. For example, \( p(6, 2) = 6 \) due to the partitions 5 + 1, 4 + 2, 4 + 1 + 1, 3 + 1 + 1 + 1, 2 + 2 + 1 + 1, 2 + 1 + 1 + 1 + 1. Write the generating function \( f(x, y) = \sum_{n,k=0}^{\infty} p(n, k)x^n y^k \) as an infinite product. Hint: For each \( m \geq 1 \) have a term in your product to account for the number of parts in the partition of size equal to \( m \).

Letting \( a_i \) be the number of copies of \( i \) in a partition, \( p(n, k) \) is the number of sequences of non-negative numbers \( a_k \) with \( a_1 + 2a_2 + 3a_3 + \ldots = n \) and the number of non-zero \( a_i \)'s equal to \( k \). Thus,

\[
    f(x, y) = \sum_{a_i} x^{a_1+2a_2+3a_3+\ldots} y^{(1 \text{ if } a_1>0)+(1 \text{ if } a_2>0)+(1 \text{ if } a_3>0)+\ldots} \\
    = \prod_{n=1}^{\infty} \sum_{a_n=0}^{\infty} x^{na_n} y^{(1 \text{ if } a_n>0)} \\
    = \prod_{n=1}^{\infty} (1 + x^n y + x^{2n} y + x^{3n} y + \ldots) \\
    = \prod_{n=1}^{\infty} \left( 1 + \frac{x^n y}{1 - x^n} \right).
\]