Exam 2 Review

Math 184
Spring 2022
Exam Details

- 4Q in 45 min
- 6 one-sided pages of notes
- Assigned Seating
Binomial Theorem and Related Identities (Chapter 4)

• Binomial Theorem
• Applications
• Generalizations
The Binomial Theorem

**Theorem:** For integers $n \geq 0$, and real numbers $x$ and $y$, we have that

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$
Multinomial Theorem

What is \((x_1+x_2+...+x_k)^n\)?

• Expand out, get a sum of terms of the form \(x_1^{a_1}x_2^{a_2}...x_k^{a_k}\).
  – Must have \(a_1+a_2+...+a_k = n\).

• What is the coefficient of this term?
  – Number of ways to pick \(x_1\) coefficient from \(a_1\) terms, pick \(x_2\) from \(a_2\) terms, etc.
  – Given by multinomial coefficient

\[
\binom{n}{a_1, a_2, \ldots, a_k} = \frac{n!}{a_1!a_2!\ldots a_k!}.
\]
Non-Integer Exponents

\((1+x)^a = 1 + ax + a(a-1)x^2/2 + ... + (a)_k x^k/k! + ...\)
Generating Functions (Chapter 8)

- Basic Idea
- Ordinary Generating Functions
  - Recurrence Relations
  - Binomial Coefficients Identities
  - Catalan Numbers
  - Partition Numbers
- Exponential Generating Functions
  - Multiplication Formula
  - “Hands” Method
  - Sterling Numbers
**Basic Method:** In order to study some interesting sequence of numbers, $a_0, a_1, a_2,\ldots$ instead turn these numbers into a single function: $f(x) = a_0 + a_1x + a_2x^2 + \ldots$ and study $f(x)$.

This function $f(x)$ is called a generating function for the sequence $\{a_i\}$. 
Generating Functions

Why is this a good idea?

1) The function $f$ can be encode the entire sequence $\{a_i\}$ with a single function.

2) Complicated combinatorial properties of the sequence $\{a_i\}$ can often be encoded as algebraic or analytic properties of $f(x)$.

3) This often lets us reduce solving complicated problems to basic algebra.
Application 1: Recurrence Relations

A recurrence relation is a way of defining a sequence of numbers by giving a formula for each element in terms of previous ones.
Basic Tools

• If two generating functions are the same, the coefficients are the same:
  If $\sum_n a_n x^n = \sum_n b_n x^n$ then $a_n = b_n$ for all $n$.

• **Geometric series:**
  $1/(1-cx) = \sum_n (cx)^n = \sum_n c^n x^n$.

• **Sums of Generating Functions:**
  $(\sum_n a_n x^n) + (\sum_n b_n x^n) = \sum_n (a_n + b_n) x^n$

• **Shifts:** If $F(x) = \sum_n a_n x^n$,
  $xF(x) = \sum_n a_n x^{n+1} = \sum_{n>0} a_{n-1} x^n$. 
Products of Generating Functions

Let $A$ and $B$ be sets of non-negative integers and let

$$f_A(x) = \sum_{a \in A} x^a, \quad f_B(x) = \sum_{b \in B} x^b.$$ 

Then,

$$f_A(x) \cdot f_B(x) = \sum x^{a+b} = \sum_n x^n \# \{ a \in A, b \in B : a+b = n \}.$$ 

Coefficients of the product are the number of ways to write $n$ as a sum of elements.
Partition Generating Function

An integer partition of \( n \) is a way of writing \( n \) as the sum of some number of 1s plus some number of 2s and so on.

In other words, it’s the number of ways to write \( n \) as \( a_1 + 2a_2 + 3a_3 + \ldots \) for non-negative integers \( a_k \).

Note that: \( \frac{1}{1-x^k} = \sum_a x^{ka} \).

So, \( \frac{1}{(1-x)(1-x^2)(1-x^3)\ldots} = \sum_p(n) x^n \).
General Products of Generating Functions

\[ A(x) = \sum_n a_n x^n, \]
\[ B(x) = \sum_n b_n x^n. \]
\[ A(x) \cdot B(x) = C(x) = \sum_n c_n x^n. \]

What is \( c_n \)?

\[ C(x) = (\sum_n a_n x^n)(\sum_m b_m x^m) \]
\[ = \sum_{n,m} a_n b_m x^{n+m} \]
\[ = \sum_k x^k (\sum_{n+m=k} a_n b_m) \]
\[ c_k = (\sum_{n+m=k} a_n b_m). \]
Combinatorial Interpretation

Suppose that you have objects of type-A and objects of type-B. Each has a size which is a non-negative integer, and there are $a_n$ objects of type-A of size $n$, and $b_m$ objects of type-B of size $m$.

Then $c_k$ is the number of ways to find a pair of an object of type-A and an object of type-B where the sum of the sizes is $k$. 
Useful Identity

\[(1 - x)^{-m-1} = \sum_{n} \binom{n + m}{m} x^n.\]

Can be used to express generating functions with polynomials in \(n\) as coefficients.
Catalan Numbers

**Definition:** The $n$th Catalan Number $C_n$ is the number of up-left lattice paths from $(0,0)$ to $(n,n)$ that stay on or above the line $x = y$.

**Note:** Catalan numbers count many other things including matching parentheses sequences.
Recursion

\[ C_n = \sum_{k=1}^{n} C_{k-1}C_{n-k}. \]
Define the generating function:

\[ H(x) = \sum_n C_n x^n. \]

\[ H^2(x) = \sum_n \left( \sum_k C_k C_{n-k} \right) x^n \]

\[ = \sum_n C_{n+1} x^n. \]

\[ xH^2(x) = \sum_n C_{n+1} x^{n+1} = H(x) - 1. \]
Quadratic Formula

We have
\[ xH^2(x) - H(x) + 1 = 0. \]

What is \( H(x) \)?

\[
H(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x}.
\]

**Note:** Only the \(-\) term makes sense, since \( H(0) \) needs to be finite.
Coefficients

Recall:

\[
\sqrt{1 - 4x} = 1 - 2x - 2x^2\binom{2}{1}/2 - 2x^3\binom{4}{2}/3 - 2x^4\binom{6}{3}/4 - \ldots
\]

So,

\[
\frac{1 - \sqrt{1 - 4x}}{2x} = 1 + x\binom{2}{1}/2 + x^2\binom{4}{2}/3 + x^3\binom{6}{3}/4 + \ldots
\]

Therefore,

\[
C_n = \frac{1}{n + 1} \binom{2n}{n}.
\]
Composition of Generating Functions

Suppose:

• \( F(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \)
• \( G(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots \)
• \( F(G(x)) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots \)

What is the combinatorial interpretation of the \( c_i \)'s?

**Note:** We almost always want to take \( b_0 = 0 \) for this. Otherwise, \( F(G(x)) \) isn’t necessarily defined even at \( x = 0 \).
Composition of Generating Functions

\[ F(G(x)) = a_0 + a_1 G(x) + a_2 G^2(x) + a_3 G^3(x) + \ldots \]

Let \( G^k(x) \) have coefficients given by

\[ G^k(x) = \sum_n c_{n,k} x^n. \]

We then have that

\[ c_n = \sum_k a_k c_{n,k}. \]

What is the combinatorial interpretation of the \( c_{n,k} \)?
Powers of Generating Functions

Note $G^k(x)$ is a product of generating functions. Define a B-structure as a thing where there are $b_i$ B-structures of size $i$. Then $c_{n,k}$ is the number of ways to find an ordered list of $k$ B-structures whose total size is $n$. 
Powers to Compositions

So we have an interpretation of $c_{n,k}$. What about $c_n = \sum_k a_k c_{n,k}$?

Define an A-structure on a set to be a thing where there are $a_k$ ways to build an A-structure on a set of size $k$.

Then we have that $c_n$ is the number of ways to find an ordered list of B-structures of total size $n$ and then build an A-structure on top of them.
Exponential Generating Functions

**Ordinary generating functions:**
Store terms in a sequence as coefficients in a power series: \( \{a_n\} \leftrightarrow \Sigma_n a_n x^n \).

Why do we need to multiply \( a_n \) just by \( x^n \)?

**Exponential generating functions:**
Store terms in a sequence as coefficients in a power series: \( \{a_n\} \leftrightarrow \Sigma_n a_n x^n / n! \).
Comparison

Ordinary:
• $F(x) = \sum_n a_n x^n$.
• $a_n = 1$, $F(x) = 1/(1-x)$.
• $a_n = n$, $F(x) = x/(1-x)^2$.
• $\sum_n a_{n-1} x^n = x F(x)$.
• Converges only if $a_n$ grows at most exponentially.

Exponential:
• $F(x) = \sum_n a_n x^n/n!$.
• $a_n = 1$, $F(x) = e^x$.
• $a_n = n$, $F(x) = xe^x$.
• $\sum_n a_{n+1} x^n/n! = F'(x)$.
• Converges more generally.
Multiplication of Exponential Generating Functions

- Multiplication of ordinary generating functions important.
- Multiplication of exponential generating functions a bit different.
- This difference is one of the most important distinguishing features.
Multiplication of Exponential Generating Functions

\[ A(x) = \Sigma_n a_n \frac{x^n}{n!} \]
\[ B(x) = \Sigma_n b_n \frac{x^n}{n!} \]
\[ C(x) = A(x)B(x) = \Sigma_n c_n \frac{x^n}{n!} \]

What is \( c_n \)?
Multiplication Formula

\[
C(x) = A(x)B(x) = \left( \sum_{m=0}^{\infty} \frac{a_m x^m}{m!} \right) \left( \sum_{k=0}^{\infty} \frac{b_k x^k}{k!} \right)
\]

\[
= \sum_{m,k=0}^{\infty} a_m b_k x^{m+k} / (m!k!)
\]

\[
= \sum_{n=0}^{\infty} x^n \left( \sum_{m+k=n} a_m b_k / (m!k!) \right)
\]

\[
= \sum_{n=0}^{\infty} x^n / n! \left( \sum_{m+k=n} \binom{n}{k} a_m b_k \right).
\]

\[
c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}.
\]

Difference from ordinary formula
Combinatorial Interpretation

Define A-structure a thing so that there are $a_n$ A-structures on a set of size $n$.

Define B-structure a thing so that there are $b_n$ B-structures on a set of size $n$.

Ordinary generating function multiplication talks about the number of ways to find an A-structure and a B-structure of total size $n$.

Exponential generating function multiplication has $c_n = \text{number of ways to partition } [n] \text{ into two sets and put an A-structure on one and a B-structure on the other}$.

If A-structure of size $k$, $\binom{n}{k}$ ways to partition $[n]$, $a_k$ A-structures and $b_{n-k}$ B-structures.
Powers of Generating Functions

Let \( A(x) = a_1x + a_2x^2/2 + a_3x^3/6 + \ldots \)

\[
A(x)^k/k! = \sum_{n=0}^{\infty} \frac{x^n}{n!} \left( \sum_{m_1+m_2+\ldots+m_k=n} \binom{n}{m_1, m_2, \ldots, m_k} a_{m_1} a_{m_2} \cdots a_{m_k} \right) / k!
\]

Define A-structure so that there are \( a_m \) A-structures on a set of size \( m \).

Number of partitions of \([n]\) into \( k \) nonempty sets with an A-structure on each.

Number of partitions of \([n]\) into \( A_1, A_2, \ldots, A_k \) with \(|A_i| = m_i\) and put A-structure on each.

Number of partitions on \([n]\) into \( A_1, A_2, \ldots, A_k \) and put an A-structure on each.
Composition of Generating Functions

So if $A(x) = a_1 x + a_2 x^2/2! + a_3 x^3/3! + ...$ and $B(x) = b_0 + b_1 x + b_2 x^2/2! + b_3 x^3/3! + ...$

What is $B(A(x))$?

It equals $b_0 + b_1 A(x) + b_2 A(x)^2/2! + b_3 A(x)^3/3! + ...$

The $x^n/n!$ coefficient is:

$b_1$ times the number of partitions of $[n]$ into one part with an $A$-structure plus

$b_2$ times the number of partitions of $[n]$ into two parts with an $A$-structure on each plus

$b_3$ times the number of partitions of $[n]$ into three parts with an $A$-structure on each plus ...
Composition of Generating Functions

So the $x^n/n!$ coefficient of $B(A(x))$ counts the number of ways to partition $[n]$ into subsets, put an $A$-structure on each subset, and put a $B$-structure on the collection of subsets.
Total Number of Permutations

So how many permutations of \([n]\) are there?

• **A-structure** is a cycle.
• **B-structure** is just a set (just partition into any number of cycles)
  
  \[
  B(x) = 1 + x + \frac{x^2}{2} + \ldots = e^x.
  \]

Generating function for number of permutations

\[
B(A(x)) = e^{\log\left(\frac{1}{1-x}\right)} = \frac{1}{1-x}.
\]

\[
\frac{1}{1-x} = \sum_n x^n = \sum_n n!(x^n/n!).
\]

There are \(n!\) permutations of \([n]\).
Sterling Number Generating Function

**Theorem:**

\[
\sum_{n,k=0}^{\infty} \frac{c(n, k)x^n}{n!y^k} = \left( \frac{1}{1-x} \right)^y.
\]