Question 1 (Counting Unions, 25 points). *How many positive integers less than 100 are either divisible by 3 or have 3 as their first digit?*

Let A denote the set of such numbers that are divisible by 3 and B the set of such numbers that have 3 as their first digit. We have that $|A| = 99/3 = 33$. We have that B contains 3 and the numbers from 30 to 39, so $|B| = 11$. The intersection is $\{3, 30, 33, 36, 39\}$, which has size 5. Thus by Inclusion-Exclusion:

$$|A \cup B| = |A| + |B| - |A \cap B| = 33 + 11 - 5 = 39.$$

Question 2 (Permutation Square, 25 points)**.** *Suppose that* π *is the permutation of* [9] *whose canonical cycle notation (without the parentheses) is* 613482957*. What is the canonical cycle notation for* π^2*?*

The cycle structure of π is (6134)(82)(957). Squaring we find that π^2 sends $1, 2, 3, 4, 5, 6, 7, 8, 9$ to $4, 2, 6, 1, 9, 3, 5, 8, 7$ respectively. The cycle structure is (63)(14)(8)(2)(975). Putting this in canonical cycle notation we get (2)(41)(63)(8)(975).

Question 3 (Binomial Coefficient Bound, 25 points)**.** *Show that for all $n \geq 0$ that*

$$\binom{3n}{n} \leq (27/4)^n.$$

Hint: use induction and consider the ratio of $\binom{3(n+1)}{n+1}$ to $\binom{3n}{n}$.

We prove this by induction on n. For $n = 0$, we note that $\binom{3n}{n} = 1 = (27/4)^n$. For the inductive step, we assume that

$$\binom{3n}{n} \leq (27/4)^n.$$

We note that

$$\begin{aligned}\binom{3(n+1)}{n+1} &= \binom{3n}{n}\left(\frac{(3n+3)(3n+2)(3n+1)}{(n+1)(2n+2)(2n+1)}\right)\\ &\leq (27/4)^n(27/4)\frac{(n+1)(n+2/3)(n+1/3)}{(n+1)(n+1)(n+1/2)}\\ &\leq (27/4)^{n+1}.\end{aligned}$$

This completes our inductive step and proves our result.

Question 4 (Set Partitions with Same Parts, 25 points)**.** *Let S be a collection of set partitions of $[n]$. Show that if the size of S is greater than 2^{n-1}, that it must contain two partitions a and b which share a part in common.*

We note that if more than one partition is $\{1, 2, 3, \ldots, n\}$ then this part shows up more than once. Otherwise, each other partition in our collection has at least 2 parts, and the total number of parts of partitions in our collection is therefore at least $2^n + 1$. Since there are only 2^n subsets of $[n]$, the pigeonhole principle implies that some subset must show up as a part in more than one of our partitions. This completes the proof.