Exam 1 Review

Math 184
Exam Format

- In class
- 4Qs in 45 minutes
- 6 one-sided pages of notes
- Randomized assigned seating
Study Options

• Lecture podcasts
• Textbook
• Problem archive
• Lecture archive
• Piazza/discord
• Office hours
Office Hours

**Daniel Kane:** Wednesday 11:15-12:45, Friday 2:15-3:45 (or by appointment) in AP&M 7131

**Philip Lamkin:** 12:00-1:00 or 2:00-3:00 on Monday and 12:00-1:00 on Friday in HSS 5012.

**Catherine Wahlenmayer:** Tuesdays 10:00-11:00 and Thursdays 1:00-3:00 in HSS 5086. This week at: [https://ucsd.zoom.us/j/95015110089](https://ucsd.zoom.us/j/95015110089)
Mathematical Induction (Ch 2)

• Induction Principle
• Strong Induction
• How to organize an inductive proof
• Things to look out for
Mathematical Induction

Suppose that you have some sequence of statements that you want to prove:

\[ S_1, S_2, S_3, S_4, \ldots, S_n, \ldots \]

The principle of mathematical induction states that it is enough to show:

1) \( S_1 \) is true.
2) For all \( n \) where \( S_n \) is true, so is \( S_{n+1} \).

Together you get \( S_1 \), which implies \( S_2 \), which implies \( S_3 \), and so on.
Format for Inductive Proof

• Give statement you want to prove by induction.
• State that you will prove it by induction on <inductive variable>.
• State and prove the base case.
• Inductive Step
  – State you are starting inductive step
  – State the inductive hypothesis
  – Use to prove next step
• Conclude your original claim
Strong Induction

For this we need **strong induction**. In order to prove $S_n$ for all $n$ we need:

1) $S_1$ is true.

2) If $S_m$ is true for all $m < n$, then $S_n$ is true.

This gives us $S_1$, which implies $S_2$. Together they imply $S_3$, and then $S_4$ and so on.
Format for **Strong** Inductive Proof

- Give statement you want to prove by **strong** induction.
- State that you will prove it by **strong** induction on <inductive variable>.
- State and prove the base case. (Or skip)
- Inductive Step
  - State you are starting inductive step
  - State the inductive hypothesis (for all smaller m)
  - Use to prove next step
- Conclude your original claim
Picking Correct Inductive Hypothesis

Usually proving a stronger statement is harder. When proving by induction this isn’t always the case. A stronger inductive hypothesis gives you more to work with when proving the inductive step. Often setting up your induction correctly makes proof much easier.
Pigeonhole Principle (Ch 1)

- Pigeonhole Principle
- Generalized Pigeonhole Principle
- Applications
Pigeonhole Principle

**Theorem:** Given $n$ pigeons each assigned to one of $m$ holes for some $m < n$, there must be some hole with at least two pigeons.
Things to Keep in Mind When Setting up Pigeonhole

- What are your pigeons?
- What are your holes?
- How are pigeons assigned to holes?
- How do you show that there are more pigeons than holes?
Generalized Pigeonhole

**Theorem:** Given $n$ pigeons each assigned to one of $m$ holes for some $m$ with $(k-1)m < n$, then there must be some hole with at least $k$ pigeons.
Basic Counting Principles (Ch 3)

- Addition Rule
- (Generalized) Multiplication Rule
- Exponent Rule
- Counting in Two Different Ways
- Multinomial Coefficients
Addition Rule

If you have two disjoint collections of items, the total number of items is the sum of the numbers from each collection.

If $S$ and $T$ are disjoint sets, then $|S \cup T| = |S| + |T|$. 
Multiplication Rule

Given two sets S and T, the number of ways to select one element from each is the product of their sizes.

$$|S \times T| = |S| \cdot |T|.$$

Exponent Rule

The number of $k$ letter words using an $n$ letter alphabet is $n^k$. 
Example: Subsets

The number of subsets of a set $S = \{x_1, x_2, \ldots, x_n\}$ of size $n$ is $2^n$. 
Example: Functions

The number of functions $f$ from a set $X$ to a set $Y$ is $|Y|^{|X|}$. 
Generalized Multiplication Rule

If you want to pick pairs of objects and:

• Have n possibilities for the first object.

• For each first object, have m possibilities for the second.
  – Note: the options might depend on which first object you picked so long as the number doesn’t.

• The total number of possible combinations is nm.
Strings without Repeats

Suppose you have an n letter alphabet and want to write a word with k letters no two of which are the same. How many ways can you do this?

• n possibilities for the first letter.
• (n-1) possibilities for the second letter.
• (n-2) possibilities for the third letter.
• ...
• (n-k+1) possibilities for the last letter.

Total = (n)(n-1)(n-2)...(n-k+1)
Falling Factorials

This is called a **Falling Factorial**.

\[(n)_k = (n)(n-1)(n-2)...(n-k+1).\]

Special case: If \(n = k\), we get
\[n! = (n)(n-1)(n-2)...(2)(1).\]

Note: \((n)_k = n! / (n-k)!\)
Multinomial Coefficients

Number of ways to order $a_1$ things of type 1, $a_2$ things of type 2,... with $a_1+a_2+... = n$

Is:

$$\frac{n!}{a_1!a_2!\cdots a_m!}.$$

This is called a **multinomial coefficient**.

$$\binom{n}{a_1,a_2,\ldots,a_m}$$
Binomial Coefficients

**Definition:** The **Binomial Coefficient** \( \binom{n}{k} \) is given by

\[
\binom{n}{k} := \frac{n!}{k!(n-k)!}.
\]

It is the number of ways to pick \( k \) items from a set of \( n \) items.
Facts about Binomial Coefficients

\[
\binom{n}{0} = 1 \quad \binom{n}{k} = \binom{n}{n-k}
\]

\[
\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}
\]
Balls and Bins (Ch 5)

- Distinguishability
- Compositions
- Stars and Bars
- Set Partitions
- Integer Partitions
Balls and Bins

In this chapter, we try to answer one question: How many ways can you place $n$ balls into $k$ bins?

Fortunately, there are many different ways to interpret this question.
Distinguishability

A more interesting consideration is that of distinguishability.

Are the balls all identical, or can you tell them apart?

Are the bins all identical, or can you tell them apart?
Distinguishable Balls into Distinguishable Bins

Each ball $b_1, b_2, \ldots, b_n$ can go into any of the $k$ bins.

Have $k$ options for each, total number of possibilities is $k^n$. 
Indistinguishable Balls into Distinguishable Bins

With indistinguishable balls, you can no longer tell where the “first” ball went. What can you tell?

You can count the number of balls in each bin.

1. 5 balls
2. 2 balls
3. 4 balls
Compositions

**Definition:** A composition (weak composition) of \( n \) is a sequence of positive (non-negative) integers \( a_1, a_2, \ldots, a_k \) so that \( a_1 + a_2 + \ldots + a_k = n \).

If we put \( n \) balls into \( k \) bins, letting \( a_i \) be the number of balls in the \( i^{th} \) bin, we get a (weak) composition of \( n \) into \( k \) parts.
Stars and Bars

In order to count compositions, we need a clever way of writing them.

• For each $i$, write $a_i$ many *’s.
• Separate these terms with a $\mid$.

So $5+2+4$ becomes:

*****|**|****
Stars and Bars and Compositions

So what happens if we write a weak composition of n into k parts in terms of stars and bars?

We get an arrangement with:
• n total stars.
• k-1 total bars.

And any such arrangement gives rise to a composition.
Counting

So we have that:
#{arrangements of n indist. balls into k dist. bins} = #{Weak compositions of n into k parts} = #{Stars and Bars arrangements with n stars and k-1 bars} = (n+k-1)Cn
(because we have n+k-1 locations and need to fill them with n stars and k-1 bars)
Non-Empty Bins

Number of compositions of n into k parts equals the number of weak compositions of n-k into k parts equals \((n-1)C(k-1)\).

Total number of compositions of n is \(2^{n-1}\).
Labeled Balls into Unlabeled Bins

- Balls labeled 1,2,...,n.
- You can tell which set of balls are in each bin.

\[
\{1,3,4\} \cup \{2,6\} \cup \{5\}.
\]

Note: \(\{1,2,3,4,5,6\} = \{1,3,4\} \cup \{2,6\} \cup \{5\}\).
Set Partitions

**Definition:** A Set Partition of a set $S$ is a collection of non-empty subsets $S_1, S_2, ..., S_k$, so that each element of $S$ is in exactly one of the $S_i$.

**Note:** The ways to put $n$ labeled balls into $k$ non-empty unlabeled bins correspond exactly to the partitions of $\{1, 2, ..., n\}$ into $k$ subsets.
Sterling Numbers

**Definition:** The Sterling numbers of the second kind are given by:

\[ S(n,k) = \# \text{ of partitions of } [n] \text{ into } k \text{ parts.} \]
Basic Facts

• $S(n,n) = 1$ [only partition is \{1\}{2}...{n}]  
• $S(n,1) = 1$ [only partition is \{1,2,...,n\}]  
• $S(0,0) = 1$ [By definition]  
• $S(n,n-1) = \binom{n}{2}$  
• $S(n,2) = 2^{n-1} - 1$
Recursion

**Theorem:** \( S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k) \).
Summation Formula

**Theorem:**

\[ x^n = \sum_k S(n,k) (x)_k. \]
Bell Numbers

**Definition:** The Bell Number $B(n)$ is the number of set partitions of a set of size $n$.

$$B(n) = \sum_{k} S(n,k).$$

$$B(n) = \sum_{k=1}^{n} \binom{n-1}{k-1} B(n-k).$$
Unlabeled Balls into Unlabeled Boxes

What can we keep track of?

- Number of balls in each box

3
1
1
3

3
1
1
2
Partitions

**Definition:** An integer partition of a positive integer $n$ is a sequence of positive integers, $a_1 \geq a_2 \geq \ldots \geq a_k$ so that $a_1 + a_2 + \ldots + a_k = n$. The $a_i$ are called the parts of the partition.

Ways of putting $n$ unlabeled balls into nonempty boxes correspond to integer partitions of $n$. 
Partition Function

**Definition:**

\( p(n) \) is the number of integer partitions of \( n \).

\( p_k(n) \) is the number of partitions of \( n \) into exactly \( k \) parts.
A **Ferrers diagram** is a way to visualize partitions. Have a grid of squares where $i^{th}$ row has $a_i$ squares.

Corresponds to partition $3+3+2+1+1 = 10$. 
Conjugates

Given a Ferrers diagram, we can come up with what is known as a **conjugate** Ferrers diagram, by flipping rows and columns.

```
3 3 2
3 3 2
2
1
1
```

```
5
3
2
```
Conjugate Properties

• The size of a partition and its conjugate are the same.
• The conjugate of the conjugate of a partition is the original partition.
• The number of parts is a partition equals the size of the largest part of its conjugate.
Consequence

**Theorem:**

\[ p_k(n) = \text{the number of partitions of } n \text{ with largest part of size } k. \]
Self-Conjugate Partitions

**Definition:** A partition is called **self-conjugate** if it is the same as its conjugate.
L Decomposition

The squares in the Ferrers diagram for any self-conjugate partition can be decomposed into Ls.

• Each L has an odd number of squares (self-conjugate).
• Each L has fewer squares than the outer one (they must nest).
Result

**Theorem:** The number of self-conjugate partitions of n is the same as the number of partitions of n into distinct, odd parts.
Lemma: Every positive integer $n$ has a unique partition into distinct powers of 2.
Theorem: For any positive integer \( n \),
The number of partitions of \( n \) into distinct parts equals the number of partitions of \( n \) into odd parts.
Odd Parts -> Distinct Parts

1) Group together and count parts of same size.

2) Write number of parts of each size in binary.

3) Take each power of 2 from binary representation to get new part size.

\[7+5+5+1+1+1+1+1+1\]

\[1 \times 7 + 2 \times 5 + 6 \times 1\]

\[(2^0) \times 7 + (2^1) \times 5 + (2^2 + 2^1) \times 1\]

\[7 + 2^1 \cdot 5 + 2^2 \cdot 1 + 2^1 \cdot 1 = 7 + 10 + 4 + 2\]
### Summary of Balls Into Bins

<table>
<thead>
<tr>
<th>Labeled Bins</th>
<th>Unlabeled Bins</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>n into k</strong></td>
<td><strong>(n+k−1)</strong></td>
</tr>
<tr>
<td>could be</td>
<td>could be</td>
</tr>
<tr>
<td>empty</td>
<td>empty</td>
</tr>
<tr>
<td><strong>n into k</strong></td>
<td><strong>(n−1)</strong></td>
</tr>
<tr>
<td>not empty</td>
<td><strong>p(n)</strong></td>
</tr>
<tr>
<td></td>
<td><strong>p_k(n)</strong></td>
</tr>
</tbody>
</table>

- **Labeled Balls**
  - \( n^k \)
  - \( \sum_{m=1}^{n} m! S(n, m) \)
  - \( k! S(n, k) \)

- **Unlabeled Balls**
  - \( 2n−1 \)
  - \( \sum_{m=1}^{k} p_m(n) \)
  - \( p(n) \)
Permutations and Cycle Structure (Ch 6)

- Permutations
- Cycles
Permutations

**Definition:** A permutation of \([n]\) is a list of the numbers from 1 to \(n\) with each number appearing exactly once. Looking at this positions of each number, this is equivalent to a bijection \(\pi:[n] \rightarrow [n]\).
Composition

Thinking of permutations as functions, we can compose two permutations of $[n]$ to get another.
Repeated Application

**Lemma:** For such m and \( \pi \), there is a \( k \) so that 
\[ \pi^k(m) = m. \]
Cycles

If \( k \) is the smallest integer so that \( \pi^k(m) = m \), we have that \( m, \pi(m), \pi^2(m),..., \pi^{k-1}(m) \) are all distinct and \( \pi^k(m) = m \).

The sequence \( m, \pi(m),..., \pi^{k-1}(m) \) is called a cycle.
Cycles

\[ \pi^k(m) \rightarrow \pi^{k-1}(m) \rightarrow \ldots \rightarrow \pi^2(m) \rightarrow \pi(m) \rightarrow m \]

\[ \pi \rightarrow \pi \rightarrow \pi \rightarrow \pi \]
Cycles

Given any permutation \( \pi \) of \([n]\), we can partition \([n]\) into cycles.

**Example:** 7146352
Cycle Representation

As every element is part of some cycle, if you can specify what each cycle is, you specify the permutation.

**Example:** 7146352 is the permutation with cycles

1 → 7 → 2 → 1

3 → 4 → 6 → 5 → 3

Any permutation of [7] with these cycles must be 7146352.
Cycle Notation

To write this more conveniently:

• Write each cycle by writing the elements in order contained within parentheses.
• Write the permutation by writing each cycle in it.

**Example:** Represent 7146352 as (172)(3465).