Exam Details

• 4 Questions in 45 minutes
• Answer only what is asked for
• 6 one-sided pages of notes
• Assigned Seating
Office Hours

Daniel Kane: W 12-1, F 1:30-3:30 AP&M 7131 or by appointment

Haixiao Wang: R 9-11, AP&M 5829 F 10-12 AP&M 5218
Ji Zheng: T 3:30-6:00 AP&M 5768 R 12:30-2:00 AP&M 5829
Zehong Zhao: MT 8-9 AP&M 6446
Mathematical Induction (Ch 2)

• Induction Principle
• Strong Induction
• How to organize an inductive proof
• Things to look out for
Mathematical Induction

Suppose that you have some sequence of statements that you want to prove: $S_1, S_2, S_3, S_4,..., S_n,...$

The principle of mathematical induction states that it is enough to show:

1) $S_1$ is true.

2) For all $n$ where $S_n$ is true, so is $S_{n+1}$.

Together you get $S_1$, which implies $S_2$, which implies $S_3$, and so on.
Format for Inductive Proof

• Give statement you want to prove by induction.
• State that you will prove it by induction on <inductive variable>.
• State and prove the base case.
• Inductive Step
  – State you are starting inductive step
  – State the inductive hypothesis
  – Use to prove next step
• Conclude your original claim
Notes

- Format is helpful in signaling what you are doing and avoiding mistakes.
- Stating inductive variable necessary:
  \[ m + (m+1) + \ldots + (m+n) = \frac{(n+1)(2m+n)}{2} \]
- Induct on \( n \) vs. Induct on \( m \).
- Base case is usually easy. Should be smallest \( n \) for which you want to prove your statement.
- Induction usually helps in situations where statements for \( n \) and \( n+1 \) easy to relate.
Strong Induction

For this we need **strong induction**. In order to prove $S_n$ for all $n$ we need:

1) $S_1$ is true.

2) If $S_m$ is true for all $m < n$, then $S_n$ is true.

This gives us $S_1$, which implies $S_2$. Together they imply $S_3$, and then $S_4$ and so on.
Format for **Strong** Inductive Proof

- Give statement you want to prove by **strong** induction.
- State that you will prove it by **strong** induction on <inductive variable>.
- State and prove the base case. (Or skip)
- Inductive Step
  - State you are starting inductive step
  - State the inductive hypothesis (for all smaller m)
  - Use to prove next step
- Conclude your original claim
Pigeonhole Principle (Ch 1)

- Pigeonhole Principle
- Generalized Pigeonhole Principle
- Applications
Pigeonhole Principle

**Theorem:** Given \( n \) pigeons each assigned to one of \( m \) holes for some \( m < n \), there must be some hole with at least two pigeons.

**Proof:** Assume for sake of contradiction that each hole has at most 1 pigeon.

Number of pigeons = \( \sum \limits_{h \text{ holes}} \left[ \text{Number of pigeons in hole } h \right] \)

\( n = \text{LHS} = \text{RHS} \leq m \)

Contradiction!
Things to Keep in Mind When Setting up Pigeonhole

• What are your pigeons?
• What are your holes?
• How are pigeons assigned to holes?
• How do you show that there are more pigeons than holes?
Dirichlet’s Theorem

**Theorem:** Let \( \alpha \) be any real number and \( q \) a positive integer. There exist integers \( n \) and \( m \) with \( 0 < m \leq q \) so that

\[
|\alpha - n/m| \leq 1/mq.
\]

This is an important result about approximability by rationals.
Packing

Take \(q+1\) numbers: \(0, \{\alpha\}, \{2\alpha\}, \{3\alpha\}, \ldots, \{q\alpha\}\)

Distribute them among intervals:
\([0,1/q], (1/q,2/q], \ldots, ((q-1)/q,1]\)

Pigeonhole Principle \(\Rightarrow\) two in same interval.

These differ by at most \(1/q\).
Generalized Pigeonhole

**Theorem:** Given n pigeons each assigned to one of m holes for some m with \((k-1)m < n\), then there must be some hole with at least k pigeons.

**Proof:** Assume for sake of contradiction that each hole has at most k-1 pigeons.

\[
\text{Number of pigeons} = \sum_{\text{holes } h} \text{[Number of pigeons in hole } h]\n\]

\[n = \text{LHS} = \text{RHS} \leq (k-1)m\]

Contradiction!
Basic Counting Principles (Ch 3)

- Addition Rule
- (Generalized) Multiplication Rule
- Exponent Rule
- Counting in Two Different Ways
- Multinomial Coefficients
Addition Rule

If you have two disjoint collections of items, the total number of items is the sum of the numbers from each collection.

If S and T are disjoint sets, then $|S \cup T| = |S| + |T|$.

Basically the definition of addition.
Multiplication Rule

Given two sets $S$ and $T$, the number of ways to select one element from each is the product of their sizes.

$$|S \times T| = |S| \cdot |T|.$$  

Basically the definition of multiplication.
Exponent Rule

The number of $k$ letter words using an $n$ letter alphabet is $n^k$.

Multiply by $n$ a total of $k$ times.
Generalized Multiplication Rule

If you want to pick pairs of objects and:

• Have \( n \) possibilities for the first object.

• For each first object, have \( m \) possibilities for the second.
  
  – Note: the options might depend on which first object you picked so long as the number doesn’t.

• The total number of possible combinations is \( nm \).
Strings without Repeats

Suppose you have an n letter alphabet and want to write a word with k letters no two of which are the same. How many ways can you do this?

• n possibilities for the first letter.
• (n-1) possibilities for the second letter.
• (n-2) possibilities for the third letter.
• ...
• (n-k+1) possibilities for the last letter.

Total = (n)(n-1)(n-2)...(n-k+1)
Falling Factorials

This is called a Falling Factorial. 
\((n)_k = (n)(n-1)(n-2)...(n-k+1)\).

Special case: If \(n = k\), we get 
\(n! = (n)(n-1)(n-2)...(2)(1)\).

Note: \((n)_k = n! / (n-k)!\)
Example Bijections

How many bijections are there \( f: X \rightarrow X \)?

If \( |X| = \{x_1, x_2, \ldots, x_n\} \):

• Have \( n \) possibilities for \( f(x_1) \)
• Have \( n-1 \) possibilities for \( f(x_2) \)
  (must be different)
• ...
• Have 1 possibility for \( f(x_n) \).

Total = \( n! = |X|! \)
Generalize

Suppose we want to put in order:

• $a_1$ things of type 1.
• $a_2$ things of type 2.
• ...
• $a_m$ things of type $m$.

Adding up to $n$ in total.
Labeled Counting

If we label the things of type \( i \): \( i_1, i_2, \ldots, i_{a_i} \ldots \)

All \( n \) things are different.

Number of ways to order \( n \) things is \( n! \).
Ways to Add Labels

Given an ordering of the initial objects, how many ways can we add labels to them?

• There are $a_1!$ ways to add labels to the type 1 objects.
• There are $a_2!$ ways to add labels to the type 2 objects.
• ...

In total there are $a_1!a_2!...a_m!$ ways to add labels.
Counting In Two Ways

The number of labeled ordering is:

1) $n!$

2) $\text{[# Orderings]} \ [a_1!a_2!...a_m!]$

Thus, the number of orderings is \( \frac{n!}{a_1!a_2!\cdots a_m!} \).

This is called a multinomial coefficient.

\( \binom{n}{a_1,a_2,\ldots,a_m} \)
Binomial Coefficients

**Definition:** The *Binomial Coefficient* \( n \) choose \( k \) is given by \( \binom{n}{k} := \frac{n!}{k!(n-k)!} \).

Sometimes we will write as \( nCk \) for convenience.

It is the number of ways to pick \( k \) items from a set of \( n \) items.
Facts about Binomial Coefficients

\[
\binom{n}{0} = 1 \\
\binom{n}{k} = \binom{n}{n-k}
\]

\[
\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}
\]
Balls and Bins (Ch 5)

- Distinguishability
- Compositions
- Stars and Bars
- Set Partitions
- Integer Partitions
Balls and Bins

In this chapter, we try to answer one question: How many ways can you place $n$ balls into $k$ bins?

Fortunately, there are many different ways to interpret this question.
Empty Bins

One way of distinguishing between versions of this problem is whether or not we allow some of the bins to be empty.
Distinguishability

A more interesting consideration is that of distinguishability.

Are the balls all identical, or can you tell them apart?

Are the bins all identical, or can you tell them apart?
Distinguishable Balls into Distinguishable Bins

Each ball $b_1, b_2, \ldots, b_n$ can go into any of the $k$ bins.

Have $k$ options for each, total number of possibilities is $k^n$.

If we want the bins to be non-empty, the problem is considerably more difficult. We’ll get back to this.
Indistinguishable Balls into Distinguishable Bins

With indistinguishable balls, you can no longer tell where the “first” ball went. What can you tell?

1 ball 2 balls 3 balls
5 balls 2 balls 4 balls

You can **count** the number of balls in each bin.
Compositions

**Definition:** A composition (weak composition) of \( n \) is a sequence of positive (non-negative) integers \( a_1, a_2, \ldots, a_k \) so that \( a_1 + a_2 + \ldots + a_k = n \).

If we put \( n \) balls into \( k \) bins, letting \( a_i \) be the number of balls in the \( i^{th} \) bin, we get a (weak) composition of \( n \) into \( k \) parts.
Stars and Bars

In order to count compositions, we need a clever way of writing them.

• For each $i$, write $a_i$ many *’s.
• Separate these terms with a $|$. 

So $5+2+4$ becomes:

*****|**|****
Stars and Bars and Compositions

So what happens if we write a weak composition of $n$ into $k$ parts in terms of stars and bars?

We get an arrangement with:

• $n$ total stars.
• $k-1$ total bars.

And any such arrangement gives rise to a composition.
Counting

So we have that:

\#\{arrangements of n indist. balls into k dist. bins\}
= \#\{Weak compositions of n into k parts\}
= \#\{Stars and Bars arrangements with n stars
    and k-1 bars\}
= (n+k-1)Cn

(because we have n+k-1 locations and need to
    fill them with n stars and k-1 bars)
Non-Empty Bins

What happens if we cannot allow bins to be empty? Then we want compositions rather than weak compositions.

Can relate the two. We have $a_1, a_2, ..., a_k$ is a composition of $n$, if and only if $a_{1-1}, a_2-1, ..., a_{k-1}$ is a weak composition of $n-k$.

Number of composition of $n$ into $k$ parts equals the number of weak compositions of $n-k$ into $k$ parts equals $(n-1)C(k-1)$. 
Another Method

We can also view this using stars and bars. We need arrangements of n stars and k-1 bars where no two bars are adjacent (and no bars at the end).

Possible locations for bars

Select k-1 locations from n-1 options. \((n-1)C(k-1)\) possibilities.
Total Number of Compositions

What if we don’t restrict the number of bins?

• Must only consider non-empty, (otherwise we can just keep adding empty bins).

• Get the total number of compositions of n.

• From the above argument have n-1 positions for bars, can select any number of them.

• Number of subsets of a set of size n-1 is $2^{n-1}$. 
Labeled Balls into Unlabeled Bins

- Balls labeled 1, 2, ..., n.
- You can tell which set of balls are in each bin.

\[
\{1,3,4\} \cup \{2,6\} \cup \{5\}.
\]

Note: \{1,2,3,4,5,6\} = \{1,3,4\} \cup \{2,6\} \cup \{5\}. 
Set Partitions

**Definition:** A **Set Partition** of a set $S$ is a collection of non-empty subsets $S_1, S_2, ..., S_k$, so that each element of $S$ is in exactly one of the $S_i$.

**Note:** The ways to put $n$ labeled balls into $k$ non-empty unlabeled bins correspond **exactly** to the partitions of $\{1, 2, ..., n\}$ into $k$ subsets.
Notation

I will use \([n]\) as a shorthand for the set \(\{1,2,\ldots,n\}\).
Sterling Numbers

**Definition:** The Sterling numbers of the second kind are given by:

\[ S(n,k) = \# \text{ of partitions of } [n] \text{ into } k \text{ parts}. \]

So \( S(n,k) \) is the number of ways to put \( n \) labeled balls into \( k \) non-empty, unlabeled bins.
Basic Facts

- $S(n,n) = 1$ [only partition is $\{1\}\{2\}...\{n\}$]
- $S(n,1) = 1$ [only partition is $\{1,2,...,n\}$]
- $S(0,0) = 1$ [By definition]
Basic Facts II

• $S(n,n-1) = \binom{n}{2}$.

Any partition of $[n]$ into $n-1$ parts must have $n-2$ parts of size 1 and 1 part of size 2.
Thus, such a partition is determined by the part of size 2.
There are $\binom{n}{2}$ ways of picking that pair of elements.
Basic Facts III

• \( S(n,2) = 2^{n-1} - 1. \)

Split \([n]\) into 2 subsets.
One of them contains 1 plus some subset of \(\{2,3,...,n\}\). The other contains the rest.
We can use any subset of \(\{2,3,...,n\}\) except for the entire thing (as then the other set will be empty).
There are \(2^{n-1} - 1\) possibilities.
Recursion

**Theorem:** $S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$.

**Proof:**
A partition of $[n]$ into $k$ parts has either:
- $n$ is in a set by itself.
- $n$ is in a set with other elements.
Summation Formula

**Theorem:**

\[ x^n = \sum_k S(n,k) (x)_k. \]

We can write polynomials in multiple ways usually we write them in terms of the powers \( x^n \), but we can also write them in terms of falling factorials.

This result shows how to convert between these representations.
Proof #1: Induction

We proceed by induction on \( n \).

**Base Case:** \( n=0 \)

\[ x^n = x^0 = 1 \]

\[ \Sigma_k S(0,k) \cdot (x)_k = S(0,0)(x)_0 = 1 \cdot 1 = 1. \]

**Inductive Step:** Assume \( x^{n-1} = \Sigma_k S(n-1,k) \cdot (x)_k \).
Inductive Step

\[
\sum_{k=1}^{n} S(n, k)(x)_k = \sum_{k=1}^{n} (S(n - 1, k - 1) + kS(n - 1, k))(x)_k \\
= \sum_{k=1}^{n} S(n - 1, k - 1)(x)_k + \sum_{k=1}^{n} kS(n - 1, k)(x)_k \\
= \sum_{k=1}^{n-1} S(n - 1, k)(x)_{k+1} + \sum_{k=1}^{n-1} kS(n - 1, k)(x)_k \\
= \sum_{k=1}^{n-1} S(n - 1, k)((x)_{k+1} + k(x)_k) \\
= \sum_{k=1}^{n-1} S(n - 1, k)(x)_k((x - k) + k) \\
= x \sum_{k=1}^{n-1} S(n - 1, k)(x)_k \\
= x^n.
\]
Bell Numbers

What about the total number of set partitions of $[n]$?

**Definition:** The Bell Number $B(n)$ is the number of set partitions of a set of size $n$. 
Bell Numbers in terms of Sterling Numbers

• $B(n)$ = number of partitions of $[n]$ into any number of parts.
• $S(n,k)$ = number of partitions of $[n]$ into exactly $k$ parts.
• So

$$B(n) = \sum_k S(n,k).$$
Bell numbers can get quite large.
B(3) = 5
B(4) = 15
B(5) = 52
B(6) = 203
B(10) = 115975
It is not hard to show that
\[ n! \geq B(n) \geq (n/2)^{n/2}. \]
Recursion

Can we find a recursion for Bell numbers?

• What is the set containing $n$?
  – If size $k$, there are $(n-1)C(k-1)$ possible sets.
  – There are then $B(n-k)$ ways to partition remaining elements.

$$B(n) = \sum_{k=1}^{n} \binom{n-1}{k-1} B(n-k).$$
Unlabeled Balls into Unlabeled Boxes

What can we keep track of?
- Number of balls in each box
Sums

• Get a collection of numbers that add up to n (the total number of balls)
  3 + 1 + 1 + 3 + 2 = 10

• Unlike with compositions, this is an unordered collection of numbers.
  – 3+1+1+3+2 the same as 2+3+1+3+1.

• How do we account for this?
  – Sort the numbers: 3+3+2+1+1.
Partitions

**Definition:** An integer partition of a positive integer \( n \) is a sequence of positive integers, \( a_1 \geq a_2 \geq ... \geq a_k \) so that \( a_1 + a_2 + ... + a_k = n \). The \( a_i \) are called the parts of the partition.

Ways of putting \( n \) unlabeled balls into nonempty boxes correspond to integer partitions of \( n \).
Partition Function

Once again, there’s no clean formula, but there is a definition.

**Definition:**

\[ p(n) \text{ is the number of integer partitions of } n. \]

\[ p_k(n) \text{ is the number of partitions of } n \text{ into exactly } k \text{ parts.} \]
A Ferrers diagram is a way to visualize partitions. Have a grid of squares where $i^{th}$ row has $a_i$ squares.

Corresponds to partition $3+3+2+1+1 = 10$. 
Properties of Ferrers Diagrams

• Total number of squares equals total size of partition.
• No gaps: For any square in the diagram the square above and the square to the left are also in the diagram (unless this would go over the edge).
• One-to-one correspondence between partitions and Ferrers diagrams.
Conjugates

Given a Ferrers diagram, we can come up with what is known as a conjugate Ferrers diagram, by flipping rows and columns.
Conjugate Properties

• The size of a partition and its conjugate are the same.
  – Each has the same number of squares.
• The conjugate of the conjugate of a partition is the original partition.
• The number of parts is a partition equals the size of the largest part of its conjugate.
Self-Conjugate Partitions

**Definition:** A partition is called **self-conjugate** if it is the same as its conjugate.
L Decomposition

The squares in the Ferrers diagram for any self conjugate partition can be decomposed into Ls.

- Each L has an odd number of squares (self-conjugate).
- Each L has fewer squares than the outer one (they must nest).
Result

**Theorem:** The number of self-conjugate partitions of $n$ is the same as the number of partitions of $n$ into distinct, odd parts.
Binary Representation

• Every positive integer $n$ has a unique binary representation.

• This corresponds to writing
  \[ n = a_k \ 2^k + a_{k-1} \ 2^{k-1} + \ldots + a_0 \]
  for $a_i = 0$ or $1$.

• Ignoring the terms with $a_i = 0$, we get a unique way of writing $n$ as a sum of distinct powers of $2$. 
Result

**Lemma:** Every positive integer $n$ has a unique partition into distinct powers of 2.

**Proof:** Use binary representation.
Theorem

**Theorem:** For any positive integer $n$, the number of partitions of $n$ into distinct parts equals the number of partitions of $n$ into odd parts.

**Example:** $n = 7$

**Distinct Parts:**
- 7
- 6+1
- 5+2
- 4+3
- 4+2+1

**Odd Parts:**
- 7
- 5+1+1
- 3+3+1
- 3+1+1+1+1
- 1+1+1+1+1+1+1
Proof

We will find a bijection between the partitions into odd parts and the partitions into distinct parts.

Note: every integer can be written unique as an odd integer times a power of 2.
Odd Parts -> Distinct Parts

1) Group together and count parts of same size.
   7 + 5 + 5 + 1 + 1 + 1 + 1 + 1 + 1 + 1
   \[ 7 \cdot 1 + 2 \cdot 5 + 6 \cdot 1 \]

2) Write number of parts of each size in binary.
   \[ (2^0) \cdot 7 + (2^1) \cdot 5 + (2^2 + 2^1) \cdot 1 \]

3) Take each power of 2 from binary representation to get new part size.
   \[ 7 + 2^1 \cdot 5 + 2^2 \cdot 1 + 2^1 \cdot 1 = 7 + 10 + 4 + 2 \]
Distinct Parts -> Odd Parts

1) Write each part as a power of 2 times an odd number.
   
   \[ 10+7+4+2 = 2^1 \cdot 5 + 2^0 \cdot 7 + 2^2 \cdot 1 + 2^1 \cdot 1 \]

2) Re-write that part as an appropriate number of copies of that number.
   
   \[ 5+5 + 7 +1+1+1+1 +1+1 = 7+5+5+1+1+1+1+1+1+1 \]
## Summary of Balls Into Bins

<table>
<thead>
<tr>
<th></th>
<th>Labeled Balls</th>
<th>Unlabeled Balls</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Labeled Bins</strong></td>
<td>(n^k)</td>
<td>((n+k-1))</td>
</tr>
<tr>
<td></td>
<td>(\sum_{m=1}^{n} m!S(n, m))</td>
<td>(\binom{n+k-1}{k-1})</td>
</tr>
<tr>
<td></td>
<td>(k!S(n, k))</td>
<td>(2n-1)</td>
</tr>
<tr>
<td><strong>Unlabeled Bins</strong></td>
<td>(\sum_{m=1}^{k} S(n, m))</td>
<td>(B(n))</td>
</tr>
<tr>
<td></td>
<td>(\sum_{m=1}^{k} p_m(n))</td>
<td>(p(n))</td>
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<tr>
<td></td>
<td>(p_k(n))</td>
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</tbody>
</table>
Permutations and Cycle Structure (Ch 6)

- Permutations
- Cycles
- Permutation Representations
- Counting Problems
- Sterling Numbers
- Permutations with only Even or Odd Cycles
Permutations

**Definition:** A permutation of \([n]\) is a list of the numbers from 1 to \(n\) with each number appearing exactly once. Looking at this positions of each number, this is equivalent to a bijection \(\pi:[n] \rightarrow [n]\).
Repeated Application

**Lemma:** For any $m$ and $\pi$, there is a $k$ so that $\pi^k(m) = m$. 
Cycles

So not only does $m$ repeat itself, it is actually the first element of this sequence to repeat itself. In particular, if $k$ is the smallest integer so that $\pi^k(m) = m$, we have that $m, \pi(m), \pi^2(m), ..., \pi^{k-1}(m)$ are all distinct and $\pi^k(m) = m$.

From there the sequence repeats itself. The sequence $m, \pi(m), ..., \pi^{k-1}(m)$ is called a cycle.
Cycle Representation

As every element is part of some cycle, if you can specify what each cycle is, you specify the permutation.
Cycle Notation

To write this more conveniently:

• Write each cycle by writing the elements in order contained within parentheses.
• Write the permutation by writing each cycle in it.
Non-Uniqueness

One issue with this representation is that there are multiple ways to write the same permutation.

This is because of two things:

• A cycle can start at any position
• Cycles can be listed in any order
Canonical Cycle Notation

One way to fix this is to remove this freedom by specifying an order. One way to do this is using **canonical cycle notation**.

• Start each cycle with the largest element.
• Write cycles in increasing order of largest element.
Parentheses

**Proposition:** Every ordering of the numbers of \([n]\) corresponds to a unique permutation of \([n]\) written in canonical cycle notation. You can determine which permutation this is even without the parentheses.
Counting Problems

We are going to consider some counting problems about permutations and cycle structure.

How many permutations have various cycles of various sizes?
Permutations with Particular Cycle Structure

Suppose that we specify values of $a_1, a_2, ..., a_n$. How many permutations of $[n]$ have exactly $a_1$ cycles of length 1, $a_2$ cycles of length 2, $a_3$ cycles of length 3, etc.
Final Answer

Final answer:

\[
\frac{n!}{a_1!a_2!\cdots a_n!1^{a_1}2^{a_2}\cdots n^{a_n}}.
\]
Sterling Numbers

**Definition:** The unsigned Sterling numbers of the first kind are given by:

\[ c(n,k) = \text{#permutations of } [n] \text{ with } k \text{ cycles} \]

The Sterling numbers of the first kind are given by:

\[ s(n,k) = (-1)^{n-k}c(n,k). \]
Examples

- $c(n, n) = 1$
  - Must have $n$ 1-cycles
- $c(n, 1) = (n-1)!$
  - One $n$-cycle
- $c(n, n-1) = n(n-1)/2$
  - One 2-cycle and $n-2$ 1-cycles

$$c(n, 2) = \sum_{k=1}^{n} \frac{n!}{2k(n-k)}.$$
Recursion

**Proposition:** For \( n \geq k \geq 1 \),

\[
c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k).
\]

**Proof (Idea):**

- Count based on where the \( n^{\text{th}} \) person sits.
- Either
  - Sits at their own table.
  - Sits at another table.
Theorem: For any non-negative integer $n$

$$(x)_n = \sum_{k=1}^{n} s(n, k)x^k.$$
Proof

Proceed by induction on \( n \).

**Base Case:** \( n = 1 \)

\((x)_1 = x = s(1,1)x^1. \)

**Inductive Step:**

Assume

\[(x)_{n-1} = \sum_{k=1}^{n-1} s(n-1, k)x^k. \]

Note that: \( s(n,k) = s(n-1,k-1) - (n-1)s(n-1,k). \)
\[
\sum_{k=1}^{n} s(n, k) x^k = \sum_{k=1}^{n} \left[ s(n - 1, k - 1) - (n - 1) s(n - 1, k) \right] x^k
\]

\[
= \sum_{k=1}^{n} s(n - 1, k - 1) x^k - (n - 1) \sum_{k=1}^{n} s(n - 1, k) x^k
\]

\[
= \sum_{k=1}^{n-1} s(n - 1, k) x^{k+1} - (n - 1) \sum_{k=1}^{n-1} s(n - 1, k) x^k
\]

\[
= x(x)_{n-1} - (n - 1)(x)_{n-1}
\]

\[
= (x - (n - 1))(x)_{n-1} = (x)_{n}.
\]
Study of Individual Cycle Sizes

**Proposition:** There are exactly \((n-1)!\) permutations of \([n]\) so that \(n\) is in a cycle of length exactly \(k\) for each \(1 \leq k \leq n\).
When you write the permutation in Canonical Cycle Notation, the cycle containing n is exactly everything listed from the appearance of n onwards.

So n is in a cycle of length k if and only if n is written \( k^{th} \) from last.

There are \((n-1)!\) ways to arrange the other \( n-1 \) elements into the other \( n-1 \) spots.
Elements in Same Cycle

**Lemma:** For any \(i,j \in [n]\), exactly half of the permutations of \([n]\) have \(i\) and \(j\) in the same cycle.

**Proof:**
- By symmetry the number is the same for any \(i,j\) so we can assume that \(i=n\), \(j=1\).
- Write in CCN. Cycle of \(n\) is everything that comes after \(n\).
- \(i,j\) in same cycle iff \(j\) comes after \(i\) in CCN.
- By symmetry, this happens half the time.
**EVEN and ODD**

**Definition:** For positive integers $n$ let
- $\text{ODD}(n) =$ the number of permutations of $n$ all of whose cycles are of odd length.
- $\text{EVEN}(n) =$ the number of permutations of $n$ all of whose cycles are of even length.
**Theorem:** For any positive integer $m$, \( \text{EVEN}(2m) = \text{ODD}(2m) \).

**Proof Idea:**

- Bijective proof using CCN.
  - Find a way to match up the permutations with even cycles and permutations with odd cycles.
Bijection

• Pair up cycles
  – must be even number
• Take last element of first in pair, and add to second.
Counting

How big are these functions?

**Proposition:**

\[
\text{EVEN}(2m) = (2m-1)^2(2m-3)^2(2m-5)^2 \ldots 1^2.
\]
Summary

So we have that:

- \( \text{EVEN}(2m) = \text{ODD}(2m) = (2m-1)^2(2m-3)^2 \ldots 1^2. \)
- \( \text{EVEN}(2m+1) = 0. \)
- **Proposition:**
  \[ \text{ODD}(2m+1) = (2m+1)(2m-1)^2(2m-3)^2 \ldots 1^2. \]
Inclusion-Exclusion (Ch 7)

- Generalized Addition Rule
- Inclusion-Exclusion
- Applications
**Inclusion-Exclusion**

**Theorem:** Given any sets $S_1, S_2, \ldots, S_n$ we have that:

$$|S_1 \cup S_2 \cup \cdots \cup S_n| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} |S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_k}|.$$
Application: Sterling Numbers

**Recall:** The number of ways to put \( n \) labeled balls into \( k \) labeled, non-empty boxes is \( k!S(n,k) \).

[First pick the set partition, then pick which part goes into which box.]

This is the same as the number of surjections from a set of size \( n \) to a set of size \( k \).
Setup

Counting all functions is easy. A function fails to be a surjection if and only if some value fails to be in the image.

Let $S$ be the set of all functions $[n] \rightarrow [k]$.

Let $A_i$ be the set of functions $[n] \rightarrow [k]$ with $i$ not in the image.

$$\#\{\text{Surjections}\} = |S| - |A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_k|$$
Inclusion-Exclusion

\[ |S| = k^n. \]

How big is \( |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_m}| \)?

\[ A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_m} = \{\text{Functions } [n] \to [k] \text{ with none of } i_1, i_2, \ldots, i_m \text{ in the image} \} = \{\text{Functions } [n] \to [k] - \{i_1, i_2, \ldots, i_m}\}. \]

\[ |A_1 \cup A_2 \cup \ldots \cup A_k| = \sum_{m=1}^{k} (-1)^{m-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_m \leq k} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_m}|. \]
Putting it Together

\[ k!S(n, k) = k^n - \sum_{m=1}^{k} (-1)^{m-1} \binom{k}{m} (k - m)^n \]

\[ = \sum_{m=0}^{k} (-1)^m \binom{k}{m} (k - m)^n. \]

Rearranging, we find:

\[ S(n, k) = \sum_{m=0}^{k} (-1)^m \binom{k}{m} (k - m)^n / k! \]
Derangements

**Definition:** A derangement is a permutation with no 1-cycles.

\[ D_n = \#\{\text{Derangements of } [n]\} \]
Counting Derangements

We want to count the number of permutations \( \pi: [n] \to [n] \) satisfying a bunch of conditions:

- \( \pi(1) \neq 1 \)
- \( \pi(2) \neq 2 \)
- \( \pi(3) \neq 3 \)
- ... 
- \( \pi(n) \neq n \)

Each condition is relatively easy to deal with individually, but many at once is hard.
Counting

How big is \(|A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_k}|\)?

This is the set of permutations of \([n]\) so that:
\[\pi(i_1) = i_1, \pi(i_2) = i_2, \ldots, \pi(i_k) = i_k.\]

• These \(k\) values are fixed.
• Can use any permutation on other \(n-k\) elements.
• Total size = \((n-k)!\)
Putting it Together

\[ D_n = n! - \sum_{k=1}^{n} \frac{(-1)^{k-1}n!}{k!} = n!(1 - 1/1! + 1/2! - 1/3! + \ldots \pm 1/n!). \]