Math 184
Exam 1 Review

Fall 2026

Exam Format

- In class
- 4 Qs in 50 minutes
- 6 one-sided pages of notes
- No other written or electronic aids (other than a simple timepiece)
- Randomized assigned seating

Review Materials

- This presentation (a brief overview of topics)
- Lecture videos
- Office hours
- Textbook
- Problem archive

Mathematical Induction (Ch 2)

- Induction Principle
- Strong Induction
- How to organize an inductive proof
- Things to look out for

Mathematical Induction

Suppose that you have some sequence of statements that you want to prove:

$S_1, S_2, S_3, S_4, \ldots, S_n, \ldots$

The principle of mathematical induction states that it is enough to show:

1) S_1 is true.
2) For all n where S_n is true, so is S_{n+1}.

Together you get S_1, which implies S_2, which implies S_3, and so on.

Format for Inductive Proof

- Give statement you want to prove by induction.
- State that you will prove it by induction on <inductive variable>.
- State and prove the base case.
- Inductive Step
  - State you are starting inductive step
  - State the inductive hypothesis
  - Use to prove next step
- Conclude your original claim

Strong Induction

Sometimes we need strong induction.

In order to prove S_n for all n we need:

1) S_1 is true.
2) If S_m is true for all $m < n$, then S_n is true.

This gives us S_1, which implies S_2. Together they imply S_3, and then S_4 and so on.

Format for Strong Inductive Proof

- Give statement you want to prove by strong induction.
- State that you will prove it by strong induction on <inductive variable>.
- State and prove the base case. (Or skip)
- Inductive Step
  - State you are starting inductive step
  - State the inductive hypothesis (for all smaller m)
  - Use to prove next step
- Conclude your original claim

Pigeonhole Principle (Ch 1)

- Pigeonhole Principle
- Generalized Pigeonhole Principle
- Applications

Pigeonhole Principle

Theorem: Given n pigeons each assigned to one of m holes for some m < n, there must be some hole with at least two pigeons.

Proof: Assume for sake of contradiction that each hole has at most 1 pigeon.

$$\text{Number of pigeons} = \sum_{\text{holes } h} [\text{Number of pigeons in hole } h]$$

n = LHS = RHS ≤ m

Contradiction!

Things to Keep in Mind When Setting up Pigeonhole

- What are your pigeons?
- What are your holes?
- How are pigeons assigned to holes?
- How do you show that there are more pigeons than holes?

Dirichlet's Theorem

Theorem: Let α be any real number and q a positive integer. There exist integers n and m with $0 < m \leq q$ so that

$$|\alpha - n/m| \leq 1/mq.$$

This is an important result about approximability by rationals.

Packing Problems

The pigeonhole principle is actually useful for proving lower bounds for a bunch of packing problems.

Theorem: Given 1001 points in the unit cube, some pair have distance at most $\sqrt{3}/10$ from each other.

Proof: Split the cube into 1000, 0.1x0.1x0.1 subcubes. Some two points from same subcube.

Generalized Pigeonhole

Theorem: Given n pigeons each assigned to one of m holes for some m with (k-1)m < n, then there must be some hole with at least k pigeons.

Proof: Assume for sake of contradiction that each hole has at most k-1 pigeons.

$$\text{Number of pigeons} = \sum_{\text{holes } h} [\text{Number of pigeons in hole } h]$$

n = LHS = RHS ≤ (k-1)m

Contradiction!

Basic Counting Principles (Ch 3)

- Addition Rule
- (Generalized) Multiplication Rule
- Exponent Rule
- Counting in Two Different Ways
- Multinomial Coefficients

Counting

Counting is a major part of combinatorics with a number of applications:

- Need for pigeonhole principle
- Compute probabilities
- Estimate length of enumerations

We will start with easy problems and slowly get more complicated.

Addition Rule

If you have two disjoint collections of items, the total number of items is the sum of the numbers from each collection.

If S and T are disjoint sets, then $|S \cup T| = |S| + |T|$.

Basically the definition of addition.

Multiplication Rule

Given two sets S and T, the number of ways to select one element from each is the product of their sizes.

$|S \times T| = |S||T|$.

Basically the definition of multiplication.

Exponent Rule

The number of k letter words using an n letter alphabet is n^k.

Multiply by n a total of k times.

Example: Subsets

The number of subsets of a set $S = \{x_1, x_2, \ldots, x_n\}$ of size n is 2^n.

Proof:

- For each i, x_i is either in the subset or not (two options).
- These choices can be made independently of each other.
- Total number of options is $2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2 = 2^n$.

Example: Functions

The number of functions f from a set X to a set Y is $|Y|^{|X|}$.

Proof:

- For each of the $|X|$ elements $x \in X$, we have $|Y|$ many possible values for f(x).
- Choices are independent.
- Total is $|Y||Y||Y|\ldots|Y| = |Y|^{|X|}$.

Generalized Multiplication Rule

If you want to pick pairs of objects and:

- Have n possibilities for the first object.
- For each first object, have m possibilities for the second.
  - Note: the options might depend on which first object you picked so long as the number doesn't.
- The total number of possible combinations is nm.

Strings without Repeats

Suppose you have an n letter alphabet and want to write a word with k letters no two of which are the same. How many ways can you do this?

- n possibilities for the first letter.
- (n-1) possibilities for the second letter.
- (n-2) possibilities for the third letter.
- …
- (n-k+1) possibilities for the last letter.

Total = (n)(n-1)(n-2)…(n-k+1)

Falling Factorials

This is called a Falling Factorial.

$(n)_k = (n)(n-1)(n-2)\ldots(n-k+1)$.

Special case: If n = k, we get

$n! = (n)(n-1)(n-2)\ldots(2)(1)$.

Note: $(n)_k = n! / (n-k)!$

Flower Arranging

You have 2 red flowers, 2 blue flowers and 1 yellow flower. You want to arrange them in a row. How many color patters can you achieve?

Possible patterns:

RRBBY, RYBRB, BRYRB,…

If we had 5 <u>different</u> colors, the answer would be 5! = 120. Is that still true?

Naming Flowers

Lets consider the exact flowers.

Suppose we have:

- 1 Rose (red) – call it R_1.
- 1 Tulip (red) – call it R_2.
- 1 Iris (blue) – call it B_1.
- 1 Morning Glory (blue) – call it B_2.
- 1 Marigold (yellow) – call it Y_1.

There are 5! ways to put these 5 flowers in order.

Overcounting

This gives us the wrong answer though. Why?

This counts (for example)

$R_1R_2B_1B_2Y_1$ and $R_2R_1B_1B_2Y_1$ as different sequences even though they lead to the same color pattern: RRBBY.

How much does this overcount by?

Labellings

Given a color pattern, how many flower patterns match it?

- 2 choices for which red flower is which.
- 2 choices for which blue flower is which.
- Only 1 yellow flower.

Total: $2 \cdot 2 = 4$.

Counting Things In Two Different Ways

How many flower orders are there?

Method 1:

Number of ways of ordering 5 things is 5! = 120.

Method 2:

[# Color Orders]·[#flower orders per color order]

= 4 [# Color Orders]

These must give the same answer!

Counting Things In Two Different Ways

So we have that $120 = 4 \cdot [\#\text{ Color Orders}]$.

Thus, the number of color orders is $120/4 = 30$.

Note: Counting in two different ways, is one of the most powerful techniques in combinatorics.

Generalize

Suppose we want to put in order:

- a_1 things of type 1.
- a_2 things of type 2.
- …
- a_m things of type m.

Adding up to n in total.

Labeled Counting

If we label the things of type i: $i_1, i_2, \ldots, i_{ai} \ldots$

All n things are different.

Number of ways to order n things is n!.

Ways to Add Labels

Given an ordering of the initial objects, how many ways can we add labels to them?

- There are $a_1!$ ways to add labels to the type 1 objects.
- There are $a_2!$ ways to add labels to the type 2 objects.
- …

In total there are $a_1!a_2!...a_m!$ ways to add labels.

Counting In Two Ways

The number of labeled ordering is:

1) n!
2) [# Orderings] [$a_1!a_2!...a_m!$]

Thus, the number of orderings is $\dfrac{n!}{a_1!a_2!\cdots a_m!}$.

This is called a multinomial coefficient.

$$\binom{n}{a_1, a_2, \ldots, a_m}$$

Binomial Coefficients

Definition: The Binomial Coefficient n choose k is given by $\binom{n}{k} := \frac{n!}{k!(n-k)!}$.

Sometimes we will write as nCk for convenience.

It is the number of ways to pick k items from a set of n items.

Facts about Binomial Coefficients

$$\binom{n}{0} = 1 \qquad \binom{n}{k} = \binom{n}{n-k}$$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Balls and Bins (Ch 5)

- Distinguishability
- Compositions
- Stars and Bars
- Set Partitions
- Integer Partitions

Balls and Bins

In this chapter, we try to answer one question:

How many ways can you place n balls into k bins?

Fortunately, there are many different ways to interpret this question.

Empty Bins

One way of distinguishing between versions of this problem is whether or not we allow some of the bins to be empty.

Distinguishability

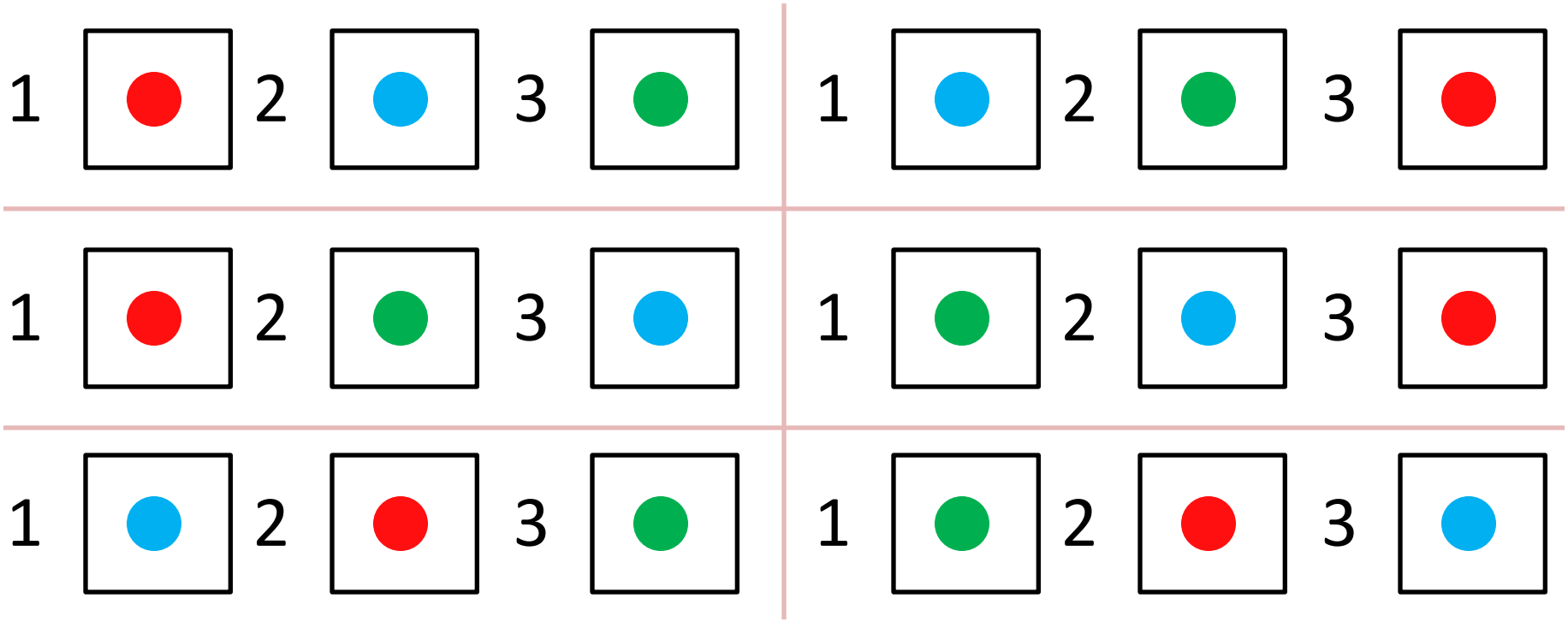
A more interesting consideration is that of distinguishability.

Are the balls all identical, or can you tell them apart?

Are the bins all identical, or can you tell them apart?

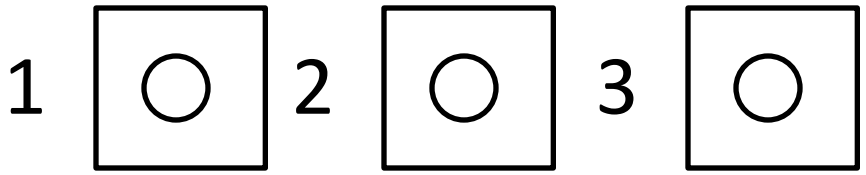
Distinguishability Example

Suppose we want to put a red, a blue, and a green ball into bins labeled 1, 2 and 3 with non empty. There are many ways to do it:

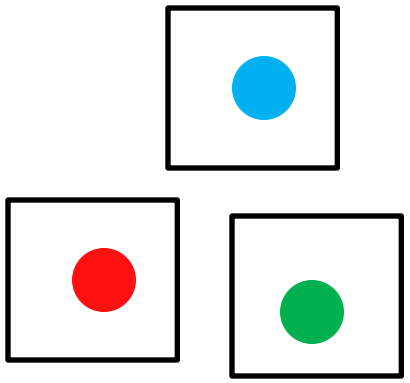


Example Continued

However, if the balls are all the same, there's only one way to do it.



Similarly, if all the bins are the same.



Cases

So…

- Balls can be distinguishable or indistinguishable.
- Bins can be distinguishable or indistinguishable.

4 cases in total.

Generally problems where things are distinguishable are easier, with distinguishable bins being more important than distinguishable balls.

Distinguishable Balls into Distinguishable Bins

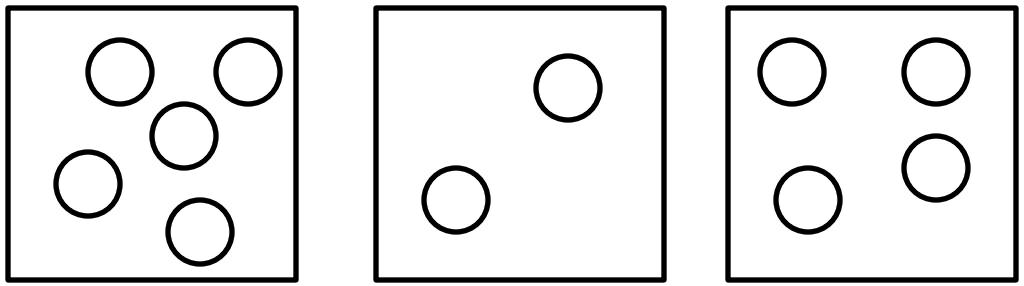
Each ball b_1, b_2,…, b_n can go into any of the k bins.

Have k options for each, total number of possibilities is k^n.

If we want the bins to be non-empty, the problem is considerably more difficult. We'll get back to this.

Indistinguishable Balls into Distinguishable Bins

With indistinguishable balls, you can no longer tell where the "first" ball went. What can you tell?



1	2	3
5 balls	2 balls	4 balls

You can count the number of balls in each bin.

Compositions

Definition: A composition (weak composition) of n is a sequence of positive (non-negative) integers $a_1,a_2,\ldots,a_k$ so that $a_1+a_2+\ldots+a_k=n$.

If we put n balls into k bins, letting a_i be the number of balls in the i^{th} bin, we get a (weak) composition of n into k parts.

Sticks and Stones

In order to count compositions, we need a clever way of writing them.

- For each i, write a_i many o's.
- Separate these terms with a |.

So 5+2+4 becomes:

ooooo|oo|oooo

Sticks and Stones and Compositions

So what happens if we write a weak composition of n into k parts in terms of | and o?

We get an arrangement with:

- n total o's.
- k-1 total |'s.

And <u>any</u> such arrangement gives rise to a weak composition.

Counting

So we have that:

#{arrangements of n indist. balls into k dist. bins}

= #{Weak compositions of n into k parts}

= #{Sticks and stones arrangements with n o's and k-1 |'s}

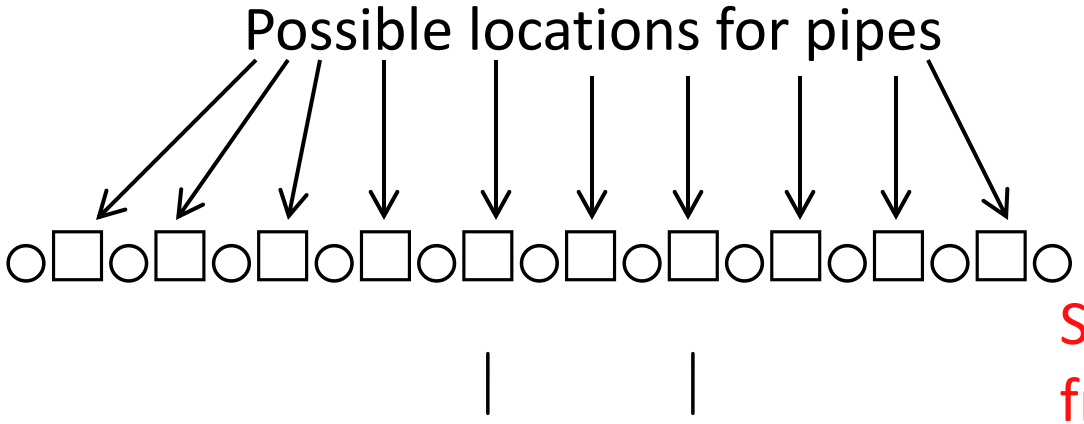
= (n+k-1)Cn

(because we have n+k-1 locations and need to fill them with n o's and k-1 |'s)

Non Empty Bins

If the bins are non-empty, have compositions.

We need arrangements of n o's and k-1 |'s where no two sticks are adjacent or at the end.



Select k-1 locations from n-1 options. (n-1)C(k-1) possibilities.

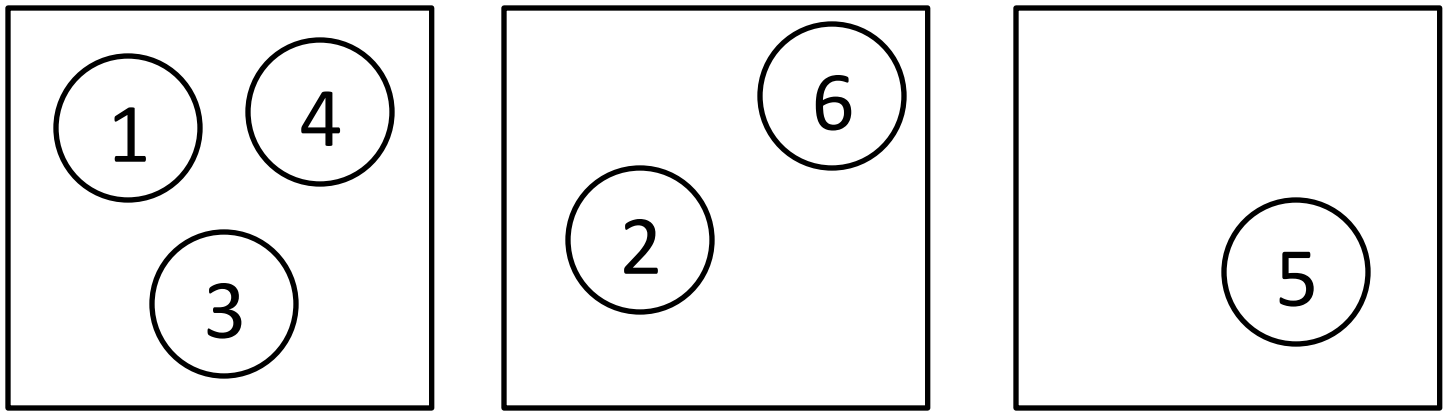
Total Number of Compositions

What if we don't restrict the number of bins?

- Must only consider non-empty, (otherwise we can just keep adding empty bins).
- Get the total number of compositions of n.
- From the above argument have n-1 positions for bars, can select any number of them.
- Number of subsets of a set of size n-1 is 2^{n-1}.

Labeled Balls into Unlabeled Bins

- Balls labeled 1,2,…,n.
- You can tell which set of balls are in each bin.



{1,3,4} {2,6} {5}

Note: {1,2,3,4,5,6} = {1,3,4} ∪ {2,6} ∪ {5}.

Set Partitions

Definition: A Set Partition of a set S is a collection of non-empty subsets $S_1, S_2, \ldots, S_k$, so that each element of S is in exactly one of the S_i.

Note: The ways to put n labeled balls into k non-empty unlabeled bins correspond exactly to the partitions of {1,2,…,n} into k subsets.

Notation

I will use [n] as a shorthand for the set {1,2,…,n}.

Counting

So we would like to count the number of set partitions of [n].

Unfortunately, there is not a simple formula for this.

What do we do? Define the problem away.

Sterling Numbers

Definition: The Sterling numbers of the second kind are given by:
S(n,k) = # of partitions of [n] into k parts.

So S(n,k) is the number of ways to but n labeled balls into k non-empty, unlabeled bins.

Is this cheating?

Kinda. But giving these numbers a name lets us prove things about them.

Basic Facts

- $S(n,n) = 1$ [only partition is {1}{2}…{n}]
- $S(n,1) = 1$ [only partition is {1,2,…,n}]
- $S(0,0) = 1$ [By definition]

Basic Facts II

- S(n,n-1) = nC2.

Any partition of [n] into n-1 parts must have n-2 parts of size 1 and 1 part of size 2.

Thus, such a partition is determined by the part of size 2.

There are nC2 ways of picking that pair of elements.

Basic Facts III

- $S(n,2) = 2^{n-1} - 1$.

Split [n] into 2 subsets.

One of them contains 1 plus some subset of {2,3,…,n}. The other contains the rest.

We can use any subset of {2,3,…,n} except for the entire thing (as then the other set will be empty).

There are $2^{n-1} - 1$ possibilities.

Recursion

Theorem: $S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$.

Proof:

A partition of [n] into k parts has either:

- n is in a set by itself.
- n is in a set with other elements.

Summation Formula

Theorem:

$$x^n = \Sigma_k S(n,k) (x)_k.$$

We can write polynomials in multiple ways usually we write them in terms of the powers x^n, but we can also write them in terms of falling factorials.

This result shows how to convert between these representations.

Bell Numbers

What about the total number of set partitions of [n]?

Once again, there's no good formula, but we can define one.

Definition: The Bell Number B(n) is the number of set partitions of a set of size n.

Bell Numbers in terms of Sterling Numbers

- B(n) = number of partitions of [n] into any number of parts.
- S(n,k) = number of partitions of [n] into exactly k parts.
- So

$$B(n) = \Sigma_k S(n,k).$$

Recursion

Can we find a recursion for Bell numbers?

- What is the set containing n?
  - If size k, there are (n-1)C(k-1) possible sets.
  - There are then B(n-k) ways to partition remaining elements.

$$B(n) = \sum_{k=1}^{n} \binom{n-1}{k-1} B(n-k).$$