More Generating Functions

Fall 2026

1 Partitions

You can develop generating functions for a lot of sequences counting integer partitions as infinite products. Each term in the product accounts for the number of parts of size k for some k.

2 Catalan Numbers

$C_n = \binom{2n}{n}/(n+1)$ Counts the number of ways to match n pairs of parentheses and a lot of other things.

3 Composition of Generating Functions

These count the number of ways to write n as a sum of any number of things. The inner generating function defines what the things involved are. The outer generating function defines how many you are allowed to use or any additional structure put on the collection.

4 Exponential Generating Functions

The same idea as ordinary generating functions except instead of associating $\{a_n\}$ to $\sum a_n x^n$, instead use $\sum a_n x^n/n!$.

Basic rules

- $c\left(\sum a_n x^n/n!\right) = \left(\sum (ca_n)x^n/n!\right)$.
- $\left(\sum a_n x^n/n!\right) + \left(\sum b_n x^n/n!\right) = \left(\sum (a_n + b_n)x^n/n!\right)$.
- $\frac{d}{dx}\left(\sum a_n x^n/n!\right) = \left(\sum a_{n+1} x^n/n!\right)$.
- $x\left(\sum a_n x^n/n!\right) = \left(\sum n a_{n-1} x^n/n!\right)$.

Furthermore, we have the product rule

$$\left(\sum a_n x^n/n!\right)\left(\sum b_n x^n/n!\right) = \left(\sum c_n x^n/n!\right)$$

where

$$c_n \sum_m \binom{n}{m} a_m b_{n-m}.$$

While the product for ordinary generating functions counts the number of ways things sum to n, this counts the number of ways to partition $[n]$ into sets with certain properties.