Exponential Generating Functions Basics

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Exponential generating functions are similar to ordinary generating function, but with some subtle differences. Here are some of the basic properties to remember:

Definition The exponential generating for a sequence $\{a_n\}$ is given by

$$\sum_{n=0}^{\infty} a_n x^n/n!$$

Basic Exponential Generating Functions The most basic exponential generating function is

$$e^x = \sum_{n=0}^{\infty} 1x^n/n!$$

Also, note that if

$$F(x) = \sum_{n=0}^{\infty} a_n x^n/n!,$$

then

$$xF(x) = \sum_{n=1}^{\infty} n a_{n-1} x^n/n!$$

This can be used to get exponential generating functions with coefficients polynomials in n.

Differentiation If

$$F(x) = \sum_{n=0}^{\infty} a_n x^n/n!,$$

then

$$F'(x) = \sum_{n=0}^{\infty} a_{n+1} x^n/n!,$$

which shifts the indices by 1. Integration can be used to shift the indices in the other direction.

Addition of Generating Functions

$$\sum_{n=0}^{\infty} a_n x^n/n! + \sum_{n=0}^{\infty} b_n x^n/n! = \sum_{n=0}^{\infty} c_n x^n/n!$$

where $c_n = a_n + b_n$.

Products of Generating Functions

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n$$

where

$$c_n = \sum_{m=0}^{n} \binom{n}{m} a_m b_{n-m}.$$

Combinatorially, if you have A-structures and B-structures that can be applied to sets, where there are a_n A-structures on a set of size n and b_n many B-structures on a set of size n, then c_n counts the number of ways to partition a set of size n into two subsets and put an A-structure on the first subset and a B-structure on the second.