Generating Functions Basics

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Generating functions are a powerful tool for proving combinatorial identities. However, they require some practice getting used to. Here are some of the basic properties to remember:

Identity A generating function is determined by its coefficients. In particular,

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$$

if and only if $a_n = b_n$ for all $n \geq 0$.

Basic Generating Functions In order to get started with writing things as generating functions, here are a few basic functions to get started with:

$$\frac{1}{(1-x)^{m+1}} = \sum_{n=0}^{\infty} \binom{n+m}{m} x^n.$$

You can also apply this substituting $x = cy^k$ for some constants c and k to get useful identities.

$$(1+x)^m = \sum_{n=0}^{m} \binom{m}{n} x^n.$$

Addition of Generating Functions

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} c_n x^n$$

where $c_n = a_n + b_n$.

Products of Generating Functions

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n$$

where

$$c_n = \sum_{m=0}^{n} a_m b_{n-m}.$$

Combinatorially, if you have classes of objects of type A and type B, where there are a_n different objects of type A of size n and b_n many objects of type B of size n, then c_n counts the number of pairs of an object of type A and an object of type B that have total size n.

So these are many of the fundamental properties of generating functions. Being able to use them effectively requires some skill with algebra and practice.