Basic Generating Functions

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1 Overview

A generating function studies a sequence $\{a_n\}$ by instead considering the function $\sum_n a_n x^n$. The idea is to use properties of the sequence to calculate the function and then use properties of the function to learn about the sequence.

2 Basic Facts

In order to compute the function, we need some basic facts to manipulate them:

- $(\sum a_n x^n) + (\sum b_n x^n) = \sum (a_n + b_n) x^n$.
- $c\,(\sum a_n x^n) = \sum (c a_n) x^n$.
- $x\,(\sum a_n x^n) = \sum a_{x-1} x^n$.
- $(1+x)^\alpha = \sum \binom{\alpha}{k} x^k$.
- $\frac{1}{1-\alpha x} = \sum \alpha^n x^n$.

3 Products

If $A(x) = (\sum a_n x^n)$ and $B(x) = (\sum b_n x^n)$ then $C(x) = A(x)B(x) = (\sum c_n x^n)$ where $c_n = \sum_m a_m b_{n-m}$.

This usefully allows one to talk about the number of ways that things can add up to n. In particular, if there are a_n many things of A-type of size n and b_n many things of B-type of size n, then c_n is the number of ways to pick a pair of a thing of A-type and a thing of B-type with total size n.

4 Applications

4.1 Recurrence Relations

If the sequence in question is defined by a linear homogeneous recurrence relation with constant coefficients, basic manipulations allow one to compute its generating function from this recurrence. This will always give a rational function and a formula for the terms of the sequence can be found using partial fractions.

4.2 Compositions

Many composition-like problems can be solved by using the rule for products of partitions and applying it on appropriate sets.

4.3 Partitions

Generating functions for functions counting various kinds of integer partitions can also often be found. This is usually as an infinite product with one term corresponding to the parts of size k for each k.