Permutations and Cycles

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1 Cycle Decomposition

A permutation can be thought of as a bijection from $[n]$ to itself. It is not hard to show that if the permutation is applied repeatedly to any element, you will eventually get back to where you started. This means that the elements of $[n]$ can be partitioned into cycles, where for each cycle the permutation sends each element to the next in a loop.

1.1 (Canonical) Cycle Notation

You can specify a permutation by listing all of its cycles. While this notation gives you many ways to represent the same permutation, canonical cycle notation fixes one of these in particular, requiring that each cycle start with its largest element and that they be listed in increasing order of largest element. Canonical Cycle Notation also has the nice property that one can still determine the permutation even if the parentheses separating where each cycle starts and ends are omitted.

2 Counting Problems

We discussed two basic counting problems relating to cycle structure:

- Count the number of permutations of $[n]$ with a specific cycle structure (i.e. the number of cycles and their sizes are all specified). For this we have an explicit formula.
- Count the number of permutations of $[n]$ with a specific number of cycles. For this we do not have a good answer, but instead define it to be given by the Sterling numbers of the first kind, which we can prove some properties of.