Problem Solving Techniques

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In this class you will often be expected to use the techniques that you learn to solve problems that aren't simply direct applications of things that you have seen. In order to do this, you will need to develop some mathematical problem solving skills. This kind of skill is very important in being able to apply your mathematical knowledge to new situations, but is hard to teach directly as by its very nature it asks you to go beyond what you were taught. A lot of these skills need to be learned through practice, but here are some general techniques to get you started.

Step 1: Read the Problem

Seriously. Read the problem carefully. If you are spending a long time working on the wrong problem you can be wasting a lot of time without making progress towards a solution. Things to check:

- Do you know the meaning of all the words in the problem statement?
- Can you restate what is being asked for in your own words?
- Can you figure out what the problem is asking for in simple examples?
- Can you list any assumptions that you are allowed to make?

Step 2: Solve the Problem

How do you actually solve a problem once you understand it? Well... usually this involves a lot of trial and error. You try a bunch of ideas, most of which won't work. Hopefully, why they don't work teaches you something about the problem and you try again. Eventually, you build up enough tools and stumble onto enough of the right ideas that you solve it. As long as you are trying out new ideas (rather than just repeating the same things over and over), you are making progress.

But what do you do if you get stuck? Well here are some basic techniques to fall back on:

- Try working out small examples or special cases.
- What techniques have we covered in class? Which of them might apply?
- Have you seen similar problems before? Can you adapt those techniques?
- Work backwards. What would allow you to solve this problem? Can you prove that? What would allow you to do that?
- Use proof by contradiction: Try to construct a counterexample. You will probably fail, but the *reason* why you fail might tell you something about why the original thing you were trying to prove is true.
- Re-read the problem statement. Are there assumptions that you aren't currently using?
- Consider extremal cases. Look at the biggest/smallest object or the best/worst case. What does this tell you about the general case?

1 Induction

Remember the general framework for an inductive proof:

- Give the statement that you are proving by induction and state that you are proving it by induction (or strong induction) and on what variable.
- State and prove the base case.
- State your inductive hypothesis (either the statement for some given n or the statement for all $m < n$ in the case of strong induction) and say that you are beginning the inductive step.
- Prove the next case of your statement given the inductive hypothesis. Give some brief statement letting the reader know that this finishes the inductive proof.

Some things to keep in mind about inductive proofs:

- Induction works by relating the instance of the problem you are working on to simpler instances of the same problem. If you want to use induction, you should think about how to take advantage of this. A lot of students make the mistake of using unnecessary induction, where their inductive proofs don't make use of the inductive hypothesis. These proofs aren't necessarily wrong, but the induction is wasteful.
- Induction arguments also often hinge critically on finding the correct inductive statement. Sometimes it is useful to pick a stronger inductive statement as this allows you to make use of a stronger inductive hypothesis. For example, if you want to show that the number of steps to solve the Towers of Hanoi problem with n disks is at most 2^n, it will actually be easier if you show that it is at most $2^n - 1$.

2 Pigeonhole Principle

The pigeonhole principle is a powerful technique for showing that you can find two things that are "close". Some things to keep in mind when trying to use it:

- What are your pigeons? What are the objects that you need to be close?
- What are your holes? You need to pick holes so that two pigeons landing in the same hole means that they are close in whatever sense you care about.
- How are you mapping pigeons to holes?
- How many pigeons/holes are there? You need to have more pigeons than holes in order for pigeonhole to apply, and many more for generalized pigeonhole.