# Homework 6 Solution

Jiaxi Nie

**Solution of Question 1.**

(a) *Given a proper schedule for the tournament, we consider the complete graph $K_n$ whose vertices are labeled by the $n$ teams, and if teams $a$ and $b$ play against each other on $k^{th}$ week, then we color edge $ab$ with color $k$. Since no team plays more than one game each week, such edge coloring on $K_n$ should be proper. So a proper schedule in $m$ weeks corresponds to an edge-coloring with at most $m$ colors. Also it is not hard to see that such correspondence is a bijection. Recall that by Vizing's Theorem, the edge chromatic number of $K_n$ is either $n-1$ or $n$, so the minimum number of weeks needed for the tournament is either $n-1$ or $n$.*

(b) *When $n$ is odd, since no team plays more than one game each week, at most $(n-1)/2$ games can be played each week. The total number of games to be played is $n(n-1)/2$, so we need at least*

$$\frac{\frac{n(n-1)}{2}}{\frac{n-1}{2}} = n$$

*weeks. When the equality holds, exactly $(n-1)/2$ games should be played each week. Therefore there will be one team sitting out each week.*

(c) *For $n$ even, $n-1$ is odd. By part (b), we know that we can arrange a tournament for the first $n-1$ teams in $n-1$ weeks, where there will be exactly one team sitting out each week. Each of the first $n-1$ team must sit out for exactly one week, since they need to play exactly $n-2$ games. Now for the $n^{th}$ team, we let it play against the sitting-out team each week. This will give us a schedule for $n$ teams in $n-1$ weeks.*

**Solution of Question 2.**

(a) *If the countries can be arbitrary disconnected sets, then any pair of countries can be adjacent(Imagine for each country there is a correspondent disc on the plane, and for each other country, there is a small area as an embassy in that disc.) In this setting, a map for $k$ countries will corresponds to a complete graph on $k$ vertices, which required $k$ colors to be properly colored.*

(b) *Given a map such that each country can have at most 2 disconnected regions. Let $G$ be a graph whose vertices are the countries, and two countries form an edge if they are adjacent. We claim that the minimum degree of $G$ is at most $11$. If this is true, we can show that $12$ colors is sufficient by induction on number of vertices $n$. For base case, when $n < 13$, this argument holds trivially. For $n \geq 13$, take a vertex $u$ with degree at most $11$, and let $G' = G - \{u\}$. By inductive assumption, we can properly color $G'$ with $12$ colors. Now add vertex $u$ back to $G'$ with such coloring. Since $u$ is connected to at most $11$ vertices in $G'$, it can be greedily colored with one of the existing $12$ colors. Hence we get a proper coloring of $G$ with $12$ colors and we are done. So it is suffices to prove the claim that minimum degree is at most $11$. Let $H$ be a graph whose vertices are regions, and two regions form an edge if they are adjacent. By definition we know that $H$ is a planar graph, so we have*

$$e(H) \leq 3v(H) - 6$$

*For any pair of adjacent regions, we can map it to the correspond pair of adjacent countries. Not hard to see that this map is surjective, so we have $e(G) \leq e(H)$. Since each country can have at most 2 regions, we have $v(H) \leq 2v(G)$. Therefore,*

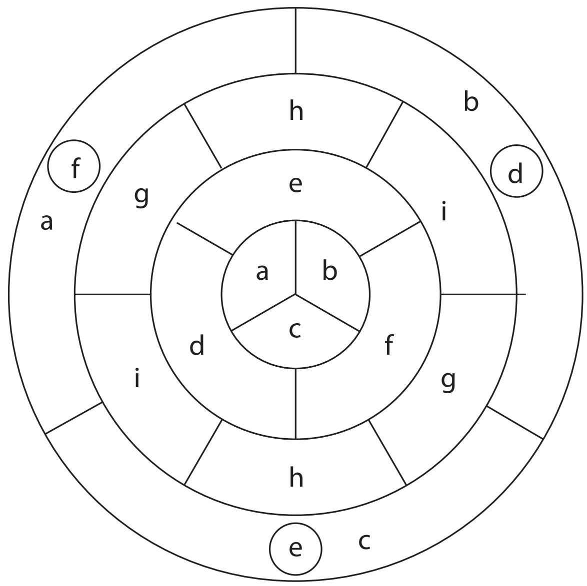
$$e(G) \leq e(H) \leq 3v(H) - 6 \leq 6v(G) - 6 < 6v(G)$$

*By hand shaking lemma we know that $\sum_{u \in V(G)} d(u) = 2e(G)$. Together with the inequality above, we have*

$$\frac{\sum_{u \in V(G)} d(u)}{v(G)} < 12$$

*This means that the average degree of $G$ is less than 12, so the minimum degree of $G$ must be less than 12, that is, at most 11. This completes the proof.*

(c) *See example below. This is a map for 9 countries $a, b, c, d, e, f, g, h, i$. Notice that any pair of them are adjacent, so 9 colors are required to color the map.*




**Solution of Question 3.** *Suppose there is a positioning of bishops on the chessboard such that at least $k$ diagonals is needed to contain all bishops, we want to show that the number of squares occupied or attacked is at least $kn - k^2$. We claim that there must be $k$ bishops such that no two of them are on the same diagonal. Consider a graph $G$ whose vertices are the diagonals, and two diagonals form an edge if there is a bishop on the intersection square of these diagonals. There is no intersection between positive slope diagonals or between negative slope diagonals. So this graph must be bipartite, where one part consists of the positive slope diagonals, and the other consists of negative slope diagonals. In this setting, a set of diagonals that contains all bishops is a vertex cover of graph $G$, and a set of bishops that share no diagonal is a matching. Since at least $k$ diagonals is needed to contain all bishops, we know that minimum vertex cover of $G$ is at least $k$. Then by Konig's Theorem, there is a matching of size $k$ in $G$, that is $k$ bishops that share no diagonal. Now we count the number of squares occupied or attacked by these $k$ bishops. Observe that if a bishop is on a diagonal with $n - i$ squares, then the other diagonal it is on will have at least $i + 1$ squares. So any bishop will occupy or attack at least $(n - i) + (i + 1) - 1 = n$ squares, here $-1$ represent the occupied square which is counted twice. Therefore,if we sum up the number of squares occupied or attacked by all bishops, we have at least $kn$. However, in this counting some squares may be double counted. Notices that for any two bishops that share no diagonal, there are exactly 2 squares that are attacked by both of them. So the number of double counted squares is exactly $2\binom{k}{2}$. So the total number of square occupied or attacked is at least*

$$kn - 2\binom{k}{2} > kn - k^2.$$