HW5 SOLUTIONS

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Q.1

(a) Let G be the planar graph (embedded in the plane) which corresponds to the polyhedron. By hypothesis each vertex has degree k, so by handshaking lemma, $\frac{vk}{2} = e := |E|$ the number of edges.

Write $\{m_1, \dots, m_l\}$ the distinct numbers in the sequence $(n_1, \dots, n_k)$ and r_i the number of occurrences of m_i in the sequence, $i = 1, \dots, l$. Let F_i be the set of faces of G that has m_i sides, and $f_i := |F_i|$. We have the set of all faces $F = \cup_i F_i$ and $f := |F| = \sum_i f_i$.

We do a double counting on number of the pairs (a, x) where $a \in V$ $x \in F_i$ and a is a vertex of x. We have

$$r_i v = m_i f_i.$$

So we have $f = \sum_i f_i = \sum_i v \frac{r_i}{m_i}$. Notice by definition of m_i and r_i, $\sum_j \frac{1}{n_j} = \sum_i \frac{r_i}{m_i}$, so $f = \sum_j v \frac{1}{n_j}$. Therefore we conclude

$$v(1 - \frac{k}{2} + \frac{1}{n_1} + \cdots + \frac{1}{n_k}) = v - e + f = 2,$$

where the last identity is by Euler.

(b) For any j, $n_j \geq 3$, hence $-\frac{1}{2} + \frac{1}{n_j} \leq -\frac{1}{6}$. By Q.1.(a), we must have $1 + \sum_{i=1}^{k}(-\frac{1}{2} + \frac{1}{n_j}) > 0$, so $k \leq 5$. As k is the number of faces we have at a vertex, $k \geq 3$. So we have three cases $k = 3, 4, 5$.

When $k = 3$: WLOG assume $n_1 \leq n_2 \leq n_3$. We proved in class that $n_1 \leq 5$. If $n_1 = 3$, look at a triangle face T, suppose the three polygon that shares edges with it are (counter-clockwisely) a-gon, b-gon and c-gon. But a, b, c takes values in $\{n_2, n_3\}$, so WLOG we must have, say, $a = b = n_2$. Then look at the vertex of T where the a-gon and b-gon share, we see the degree sequence gives $(n_1, n_2, n_3) = (3, a, b) = (3, n_2, n_2)$, which means $n_1 = n_2$. Now the condition $1 + \sum_{i=1}^{k}(-\frac{1}{2} + \frac{1}{n_j}) > 0$ becomes $\frac{2}{n_2} - \frac{1}{6} > 0$ so $n_2 \in \{3, \dots, 11\}$. Now check for all possible n_2's, we only have the following solutions: $(3, 3, 3)$, $(3, 4, 4)$, $(3, 6, 6)$, $(3, 8, 8)$, $(3, 10, 10)$.

If $n_1 = n_2 = 4$, we have infinitely many solutions $(4, 4, n)$.

If $n_1 = 4$ and $n_2 > 4$, we see $1 + \sum_{i=1}^{k}(-\frac{1}{2} + \frac{1}{n_j}) > 0$ implies $\frac{1}{n_3} - \frac{1}{20} > 0$, so $n_3 \in \{5, \dots, 19\}$, we can check all possibilities and found the following solutions: $(4, 6, 6)$, $(4, 6, 8)$, $(4, 6, 10)$.

If $n_1 = 5$, $1 + \sum_{i=1}^{k}(-\frac{1}{2} + \frac{1}{n_j}) > 0$ implies $\frac{1}{n_3} - \frac{1}{10} > 0$, we can check all possibilities and found the following solutions: $(5, 5, 5)$, $(5, 6, 6)$.

When $k = 4$: WLOG assume $n_1 \leq n_2 \leq n_3 \leq n_4$. If $n_1 = 4, 5$, $1 + \sum_{i=1}^{k}(-\frac{1}{2} + \frac{1}{n_j}) > 0$ isn't satisfied. So WLOG $n_1 = 3$. If $n_2 = n_3 = 3$, we see any n_4 gives a solution: $(3, 3, 3, n)$. If $n_3 > 3$, $1 + \sum_{i=1}^{k}(-\frac{1}{2} + \frac{1}{n_j}) > 0$ implies $\frac{1}{n_4} - \frac{1}{12} > 0$. So

$\{n_4, n_3\} \subset \{4, \dots, 11\}$, and by checking all possibilities we have the following solutions: $(3,4,4,5)$, $(3,3,5,5)$, $(3,4,4,4)$, $(3,3,4,4)$.
When $k = 5$: WLOG assume $n_1 \le n_2 \le n_3 \le n_4 \le 5$. Clearly $n_1 \ge 3$, then $1 + \sum_{i=1}^{k}(-\frac{1}{2} + \frac{1}{n_j}) > 0$ already implies $\frac{1}{n_5} - \frac{1}{6} > 0$, which means $n_5 \in \{3, 4, 5\}$. Check all the possibilities we only have solutions: $(3,3,3,3,5)$, $(3,3,3,3,4)$, $(3,3,3,3,3)$.
Finally numerically we have above solutions, in order to claim that they are indeed from a semi-regular solid, we need to construct the semi-regular solid manually. We have found 5 sequences with all number equal, they corresponds to the Platonic polyhedra. We have found two class of solutions $(4,4,n)$ and $(3,3,3,n)$, and they corresponds to prisms and antiprisms. The remaining 13 polyhedra are Archimedean solids, please check Wikipedia page "Archimedean solid" to convince yourself they really exists!

Q.2

(a) We prove by induction on $\mathrm{cr}(G)$. The base case when $\mathrm{cr}(G) = 0$, we have $0 \ge e - 3v + 6$ as a property of planar graphs proved in the lecture.
Now suppose $\mathrm{cr}(G) = k > 0$. Let f be a planar embedding of G with k crossings. Now at a point (of $\mathbb{R}^2$, not a vertex of G yet) where two edges $e_1 = \{v_1, v_2\}, e_2 = \{v_3, v_4\}$ cross, we introduce a new vertex v_0 and connects v_0 to v_1, v_2, v_3, v_4 along the trajectory of the previous edges e_1, e_2. We call the new graph G' and write $e' = |E(G')|$, $v' = |V(G')|$. We notice that $\mathrm{cr}(G') \le k - 1$. So by strong induction hypothesis, $k - 1 \ge \mathrm{cr}(G') \ge e' - 3v' + 6$. Observe that $e' = e + 2$ and $v' = v + 1$, substitute into previous inequality we have $k \ge e - 3v + 6$ which is what we want.
(b) Let $f_i : G_i \to \mathbb{R}^2$ be the planar embedding s.t. the number of crossing for each graph is $\mathrm{cr}(G_i)$. WLOG, we can translate the image of f_i s.t. their images, as sets on $\mathbb{R}^2$, are disjoint. (It should be noted here that a planar embedding of a graph also embeds the edges into the plane, we also requires the embedded edges are disjoint for distinct G_i's.) Now we consider a planar embedding $f : G \to \mathbb{R}^2$ s.t. $\forall v \in V_i$, we map $f(v) := f_i(v)$. We also wish to embed the edges, $\forall e = \{u, v\} \in E(G)$, by hypothesis on G_i being connected components, $\exists i$ s.t. $e \in E(G_i)$ so we define that f maps the edge e into the plane as how f_i maps e into the plane. By assumption that images of f_i are disjoint, we have the total number of crossings in the planar embedding f is $\sum_i \mathrm{cr}(G_i)$, hence $\mathrm{cr}(G) \le \sum_i \mathrm{cr}(G_i)$.
On the other hand, let $f : G \to \mathbb{R}^2$ be a planar embedding s.t. the number of crossings is $c = \mathrm{cr}(G)$. Now consider the restriction $f_i : G_i \to \mathbb{R}^2$, these are planar embeddings of G_i's. Write the number of crossings happened in the embedding f_i as c_i. By definition we have $\mathrm{cr}(G_i) \le c_i$. Then we have

$$\mathrm{cr}(G) = c \ge \sum_i c_i \ge \sum_i \mathrm{cr}(G_i),$$

which is what we want.
(c) Let $q = \lfloor \frac{n^2}{2m} \rfloor$. We do a division with remainder $n = qk + r$ s.t. $q, r \in \mathbb{Z}$ and $0 \le r < q$. Consider q groups of vertices $V_1, \dots, V_q$ s.t. $|V_1| = |V_2| = \cdots = |V_r| = k + 1$ and $|V_{r+1}| = |V_{r+2}| = \cdots = |V_q| = k$. Let the vertex set $V = \cup_{i=1}^{q} V_i$, clearly we have $|V| = n$ vertices. Now we place vertices each V_i on disjoint circles on the plane. We keep adding edges as straight lines inside the circle between vertices on the same circle. Also fix $v_i \in V_i$, we can add edges as curves outside the circles from each v_i to every point in $V_{i+1} \backslash \{v_{i+1}\}$ s.t. the outside edges don't cross each

other (this is possible because we can place the circles s.t. the line T tangent to the circle at v_{i+1} has V_i and $V_{i+1}\backslash\{v_i\}$ on the same side and we connects v_i to $V_{i+1}\backslash\{v_i\}$ with curves not intersecting T).
Firstly we prove it's possible to add more than m edges into V in the way described above. Indeed, the maximum number of edges that we can add to V is achieved when each V_i is a complete graph. Also there are n edges outside of the circles. So in total we have

$$\begin{aligned}
&r\frac{(k+1)k}{2}+(q-r)\frac{k(k-1)}{2}+n\\
=&rk+q\frac{k(k-1)}{2}+n=rk+(n-r)\frac{k-1}{2}+n=r\frac{k+1}{2}+n\frac{k-1}{2}+n\\
=&n\frac{k+1}{2} \quad \text{, because } r\frac{k+1}{2}\geq 0\\
=&n\frac{\lfloor\frac{n}{q}\rfloor+1}{2}>n\frac{\frac{n}{q}}{2} \quad \text{, because } k=\lfloor\frac{n}{q}\rfloor>\frac{n}{q}-1\\
\geq&n\frac{\frac{2m}{n^2}}{2} \quad \text{, because } q=\lfloor\frac{n^2}{2m}\rfloor\leq\frac{n^2}{2m}\\
=&m.
\end{aligned}$$

This means we can at least add m edges into V following our rule. We can do so and call such a resulting graph G.
Then we prove for above G, $\mathrm{cr}(G)\leq\frac{m^3}{n^2}$. The crossings only happen inside of the circles, and as each circle has at most $k+1$ vertices, we have at most $\binom{k+1}{4}$ crossings in each circle. So an upper bound for $\mathrm{cr}(G)$ is

$$\begin{aligned}
&q\binom{k+1}{4}=q\frac{(k+1)k(k-1)(k-2)}{1\cdot 2\cdot 3\cdot 4}\\
\leq&n\frac{(k+1)(k-1)(k-3)}{1\cdot 2\cdot 3\cdot 4} \quad \text{, because } qk=n-r\leq n\\
\leq&n\frac{k^3}{1\cdot 2\cdot 3\cdot 4} \quad \text{, because } (k-1)(k+1)=k^2-1.
\end{aligned}$$

Now as $q=\lfloor\frac{n^2}{2m}\rfloor$, we have $q>\frac{n^2}{4m}$ by property of floor function. So $k=\lfloor\frac{n}{q}\rfloor\leq\frac{n}{q}<\frac{4m}{n}$. Combining this inequality to above inequality, we have $\mathrm{cr}(G)<\frac{8}{3}\frac{m^3}{n^2}$.