Homework 4 Solution

Jiaxi Nie

Solution of Question 1. *If G is planar, then any subgraph of G must also be planar. Hence every block of G is planar. On the other hand, if every block of G is planar, let k be the number of blocks of G. We will show that G is planar by induction on k. When $k = 1$, there is nothing to prove. When $k > 1$, we assume that the argument holds for $k-1$. Consider the block graph of G, we know that it is always a tree. Pick a leaf of the block graph, by definition, its neighbor u must be a cut vertex of G. Removing u from G will disconnect the graph into two subgraphs A and B, where A has $k-1$ blocks, and B is a block. By inductive assumption, A must be planar. B is also planar. A and B has exactly 1 common vertex u. Since A is planar, we can properly embed A onto a sphere (properly embed means we can draw it with no crossing edge). Under this embedding, let F be a face such that u is a vertex on its sides. Remove F from the sphere, then we can map the punctured sphere onto a disk on the plane such that u is on the boundary of the disk(Imaging cut a hole on the face F of the sphere, and then stretch the sphere into a disk). So we find a planar embedding of A such that A is contained inside a disk and u is on the boundary. Same argument can also work on B. Since two disk can be draw on the plane with exactly 1 intersection point on the boundary, we can combine the embedding of A and B to get a planar embedding of G. This completes the proof.*

Solution of Question 2. *Let $x_i \in V$ be the string with 1 at position i, and 0 elsewhere, and let $W = \{x_1, x_2, \dots, x_n\}$. Removing W from v will disconnect $(0,0,\dots,0)$ and $(1,1,\dots,1)$, $|W| = n$. So by definition we have $\kappa((0,0,\dots,0),(1,1,\dots,1)) \leq n$. In order to show $\kappa((0,0,\dots,0),(1,1,\dots,1)) = n$, it suffices to show that there exist n vertex disjoint paths from $(0,0,\dots,0)$ to $(1,1,\dots,1)$). Let $x_{i,k}$ be the string with k 1's and $(n-k)$ 0's, where 1's are positioned at k consecutive positions begins at position i(If the index exceed n, then it go to position 1). Let Pi be the path $(0,0,\dots,0) \to x_{i,1} \to x_{i,2} \to \cdots \to x_{i,n-1} \to (1,1,\dots,1)$, $1 \leq i \leq n$. Then $P_1, P_2, \dots, P_n$ are n vertex disjoint paths from $(0,0,\dots,0)$ to $(1,1,\dots,1)$. In order to disconnect $(0,0,\dots,0)$ and $(1,1,\dots,1)$, the cut set must contain at least 1 vertex in each path, hence the size of the cut set is at least n. So we have $\kappa((0,0,\dots,0),(1,1,\dots,1)) \geq n$. Therefore $\kappa((0,0,\dots,0),(1,1,\dots,1)) = n$.*

Solution of Question 3. *(a) If G is bipartite, then any cycle in G will have even number of edges. For a given face in a planar embedding of G, its sides form a cycle, so the number of sides here must be even. On the other hand, if each face in a planar embedding of G has even number of sides, then let's consider an arbitrary cycle C of G. Let G' be the subgraph of G consisting C and all vertices and edges inside C. Let l be the length of C, and let $F_1, F_2, \dots, F_k$ be all faces inside C, $l_1, l_2, \dots, l_k$ be the number of their sides respectively. Notice that l is the number of sides of the infinite face of G', so by the dual hand shaking lemma, we have*

$$l + \sum_{i=1}^{k} l_i = 2|E(G')|.$$

Equivalently we have $l = 2|E(G')| - (\sum_{i=1}^{k} l_i)$. Notice that $l_1, l_2, \dots, l_k$ are all even, so we have l is even.

(b)Method 1: We color the regions in the following way: if a region is inside even number of circles, we color it black; if a region is inside odd number of region, then we color it white. Notice that if two region R_1 and R_2 are adjacent, then they must share an arc of some circle C. Without lost of generality we can assume R_1 is inside even number of circle, and R_1 is inside C. Then R_2 must be outside of C, but it is inside and only inside every other circle that contains R_1. So the number of circle that contain R_2 must be odd. Therefore, this coloring is valid.

Method 2: Consider a graph whose vertices are the regions and two regions form an edge if and only if they share an arc. It suffices to show that this graph is bipartite. Not hard to see that this graph is planar, and each face(except the infinite face) of this graph under the natural embedding corresponds to an an intersection point of circles. To go around an intersection point, we must cross each circle twice. So the number of sides of each such face is even. Then by the dual hand shaking lemma, the number of sides of the infinite face should also be even. Hence we conclude that this graph is bipartite by part (a).