HW3 SOLUTIONS

JI ZENG

Q.1

Choose any vertex v, let k be the number of copies of v in L. We recall the process of contructing a tree from a list in the proof of Cayley's theorem.
If the last vertex of L isn't v: for each time it appears in L, we find a suitable missing vertex u and add the edge $\{u, v\}$. After the last v in the list L is considered, we add v into the list of missing vertices. As every vertex, except for the last of the list, serves as a missing vertex and pairs with a vertex in L to be an added edge exactly once. We conclude there are exactly $k+1$ edges with v been added, so $\deg(v) = k+1$.
If the last vertex of L is v: again, for each time it appears in L, we find a suitable missing vertex u and add the edge $\{u, v\}$. After the last v is considered, we add v into the list of missing vertices. Since v is the last of L, there are only two missing vertices left and one of them is v. According to the algorithm, we add an edge consisting of the two last missing vertices. Hence $\deg(v) = k+1$ as well.
It's also possible to write a proof based on the process, in the proof of Cayley's theorem, of constructing a list from the tree.

Q.2

"Only if" part: Suppose we have an Eulerian circuit, we can express it as a sequence S of vertices (possibly with repetitions)

$$v_0, v_1, \dots, v_m = v_0$$

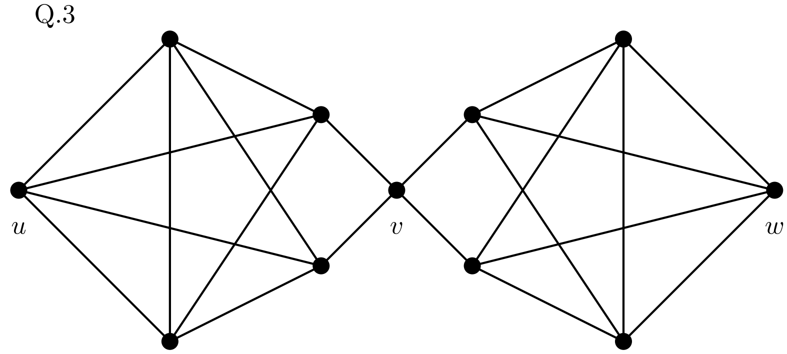
Fix an arbitrary v, for any $i \in [1, m-1]$ s.t. $v_i = v$, we see that $[v_{i-1}, v_i]$ is an edge pointing towards v (which contributes 1 to in-degree of v) and $[v_i, v_{i+1}]$ is an edge emanating from v (which contributes 1 to out-degree). And if $v_0 = v_m = v$, then $[v_{m-1}, v_m]$ is an edge pointing towards v (which contributes 1 to in-degree of v) and $[v_0, v_1]$ is an edge emanating from v (which contributes 1 to out-degree). Therefore $v \neq v_0$, then $d_{in}(v) = d_{out}(v) =$ number of copies of v in S. If $v = v_0$, then $d_{in}(v) = d_{out}(v) =$ number of copies of v in $S. - 1$.
"If" part: Suppose now that G is a connected di-graph with $d_{in}(v) = d_{out}(v), \forall v$.
Firstly we prove that for arbitrary $v \in G$ with $d_{in}(v) = d_{out}(v) \neq 0$, we can find a circuit C with $v \in C$. Indeed, consider the process of choose a sequence of points $v_0 = v, v_1, v_2, \dots$. Where v_i is chosen such that $v_{i-1} \to v_i$ is an directed edge of G which haven't been considered before. We terminate the process when we reach v_k and there's no unconsidered edge emanating from v_k. We show that such our process only terminates at $v_k = v_0$, suppose $v_k = u \neq v_0$, then let q be the number of copies of u in the sequence

$$v_0, v_1, \dots, v_k.$$

We see that we have considered q edges pointing towards u, and $q-1$ edges emanating from u, but by the condition $d_{in}(u) = d_{out}(u)$, we see that there must be at

least one unconsidered edge emanating from u, contradicting the assumption that the process terminates at $u \neq v$.
Secondly we point out that suppose we have a circuit $C \subset G$, then consider the graph $G \backslash C$ obtained by removing all the edges in C. By the same logic as the "only if" part of the proof, $d_{in}(v, C) = d_{out}(v, C)$ where the degree denotes degree in the subgraph C. So $d_{in}(v, G\backslash C) = d_{out}(v, G\backslash C)$, and we can construct circuits in $G\backslash C$. Therefore, we can find a collection $\{C_1, C_2, \ldots, C_r\}$ of circuits s.t. their union $\cup_{i=1}^{r} C_i = G$ and no pair of them share any edges.
Finally it suffices to show, by induction on r, that we can combine these circuits into one Eulerian citcuit. For a general r, as G is connected, then there must be another circuit, WLOG say it's C_r, which shares a vertex with C_1. Then we can combine the two circuit together into a larger one C_1'. Hence we reduced to the task of combining $\{C_1', C_2, \ldots, C_{r-1}\}$ into one Eulerian circuit, which can be done by induction hypothesis.

Q.3



This graph isn't Hamiltonian. Suppose on the contrary, it's Hamiltonian. Consider we express the Hamiltonian cycle C by a sequence:

$$v_0 = u, v_1, v_2, \ldots, v_n = u.$$

Let k be s.t. $v_k = w$, then $\exists i \in [1, k-1]$ s.t. $v_i = v$, as all the path connecting u and w must pass v. Similarly, $\exists j \in [k+1, n-1]$ s.t. $v_j = v$, contradicting that C is a cycle.
Other correct solutions may apply.