Homework 2 Solution

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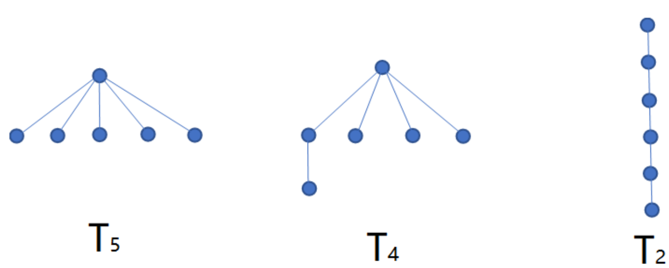
Solution of Question 1. *Let v be a vertex of T, and let $N_i(v) := \{u \in V(T) : d(u,v) = i\}$, where $d(u,v)$ is the distance between u and v in T. Consider a partition of vertices of T, $V(T) = A \cup B$, such that $A = \cup_{k\geq 0} N_{2k}(v)$ and $B = \cup_{k\geq 0} N_{2k+1}(v)$. We claim that this is the partition that makes G a bipartite graph. Otherwise, without lost of generality, suppose there exist two vertices $u, w \in A$ such that uw is an edge of G. Then by definition, there exist two paths of even length from v to u and from w to v. Connect these two paths by uw, we get a closed walk of odd length in G, which contradict the fact that G is a bipartite graph. Since G is connected, for any pair of vertices, there exist a path between them. We can determine whether they are in the same part by whether the length of the path between them is even or odd. Hence, such partition is unique.* □

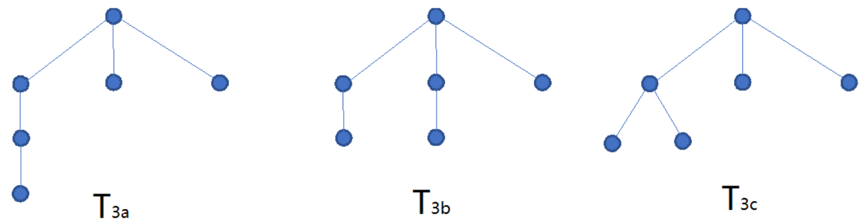
Solution of Question 2. *A tree T with 6 vertices must have exactly 5 edges. Let the six vertices be $v_1, v_2, v_3, v_4, v_5, v_6$. Then by the Handshaking Lemma, we have*

$$d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5) + d(v_6) = 10.$$

For any $1 \leq i \leq 6$, $d(v_i)$ is a positive integer. So the maximum degree $\Delta(T)$ must satisfy $2 \leq \Delta(T) \leq 5$. Assume that $d(v_1) \geq d(v_2) \geq \cdots \geq d(v_6)$, and consider the degree sequence $(d(v_1), d(v_2), d(v_3), d(v_4), d(v_5), d(v_6))$. When $\Delta(T) = 5$, the degree sequence can only be $(5,1,1,1,1,1)$, look at the vertex with largest degree and its neighborhood will determine the whole tree, and we called this kind of tree T_5. When $\Delta(T) = 4$, the degree sequence can only be $(4,2,1,1,1,1)$. The neighborhood of the vertex with degree 4 must contain the unique vertex of degree 2, otherwise, if all four of them has degree 1, they cannot connect to anything else, and the graph will hence be disconnected. So in this case the tree is also uniquely determined, denoted as T_4. When $\Delta(T) = 2$, the degree sequence can only be $(2,2,2,2,1,1)$, in this case the tree can only be a path, denoted as T_2. Trees T_5, T_4 and T_2 can be drawn as follows:



Specially, when $\Delta(T) = 3$, the degree sequence can be $(3,2,2,1,1,1)$ or $(3,3,1,1,1,1)$. For the case of $(3,2,2,1,1,1)$, consider the neighborhood of the vertex with degree 3. The number of neighbors with degree 2 is either 1 or 2(It cannot be 0, since if all three of them has degree 1, they cannot connect to anything else). If it is 1, then it must connected to another vertex with degree 2, and we get the whole tree, denoted as T_{3a}. If the number is 2, then the tree is determined, denoted as T_{3b}. When the degree sequence is $(3,3,1,1,1,1)$, consider the neighborhood of one of the vertex of degree 3. One of the vertex in this neighborhood must also have degree 3, and it is connected to other two vertices with degree 1, this give us the tree denoted T_{3c}. Trees T_{3a}, T_{3b} and T_{3c} can be drawn as follows:



Note that two graphs that are isomorphic must have the same degree sequence, so the tree above with different degree sequences are not isomorphic. To distinguish T_{3a} *and* T_{3b}*, we observe the neighborhood of the unique vertex of degree 3. If* T_{3a} *and* T_{3b} *are isomorphic, then these vertices with degree 3 must be mapped to each other, and hence their neighbors must have the same degree sequence. However, in* T_{3a}*, there is only one neighbor with degree 2, while in* T_{3b} *there are two. So* T_{3a} *and* T_{3b} *are not isomorphic.* □

Solution of Question 3. *Let* P *be the unique path from* v *to* u *in* T*, and let* m *be the minimum edge weight on this path. Not hard to see that a truck of size* m *can make it from* v *to* u *along this path* P*. We want to show that a truck of size* m *is the largest truck that can make it from* v *to* u*. Suppose for contradiction that it is possible for a truck with size larger than* m *to make it from* v *to* u*, then there exist a path* P' *from* v *to* u *not in* T*, whose minimum edge weight is larger than* m*. Let* $\{a, b\}$ *be an edge in* P *with minimum weight* m*, then* $\{a, b\}$ *is not in* P'*. Deleting* $\{a, b\}$ *from* T *will disconnect the tree into two connected components* A *and* B*, while* v *is in* A *and* u *is in* B*. Since* P' *is a path from* v *to* u*, there must be and edge* $\{c, d\}$ *in* P' *such that* c *is in* A *and* d *is in* B*. So adding* $\{c, d\}$ *will result in a new spanning tree* T' *of* G*. Note that* $\{c, d\}$ *is in* P'*, hence its weight must be larger than* m*. So the total weight of* T' *must be larger than that of* T*. This contradict the fact that* T *is maximum.* □