HW1 SOLUTION

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Q.1

(a) We define the degree to be $d_v := \#\{\text{edges } e \text{ s.t. } v \in e\}$. Consider the set of pairs $S = \{(e, v); v \in e\}$. We apply double counting to S. On the one hand, $\forall e'$, define $S_{e'} := \{(e', v) \in S\}$ the pairs with edge e', then $|S_e| = 2, \forall e$ and $|S| = \sum_e |S_e| = 2|E|$. On the other hand, $\forall v'$, define $S_{v'} := \{(e, v') \in S\}$ the pairs with vertex v', then $|S_v| = d_v$ by our definition and $|S| = \sum_v |S_v| = \sum_v d_v$. So $\sum_v d_v = 2|E|$ is the new handshaking lemma.

(b) For any vertex v, we define d'_v to be the number of non-loop edges incident to v and d''_v to be twice the number of loops on v. And define the degree of v to be $d_v = d'_v + d''_v$ for all $v \in V$. Write E_1 to be the set of all the non-loop edges and E_2 to be the set of all the loop edges. By the normal handshaking lemma, we have $\sum_v d'_v = 2|E_1|$. We also have $\sum_v d''_v = 2|E_2|$, because every loops contributes to exactly 2 on both side. Hence $\sum_v d_v = \sum_v d'_v + \sum_v d''_v = 2|E_1| + 2|E_2| = 2|E|$.

(c) Consider the set of triples $S := \{(e, u, v); e = [u, v]\}$, where $[u, v]$ denotes the ordered pair (a.k.a directed edge) from u to v. We count the size of S in three ways. Firstly, $\forall e'$, there exists a unique triple (e', u, v) in S containing e' so $|S| = |E|$. On the other hand, $\forall u' \in V$, consider $S_{u'} = \{(e, u', v) \in S\}$ the triples having the second term as u', by definition of out-degree, we have $|S_u| = d_{out}(u), \forall u$. So $|S| = \sum_{u \in V} d_{out}(u)$. Finally for all v', consider $S_{v'} = \{(e, u, v') \in S\}$ the triples having the third term as v', by definition of in-degree, we have $S_v = d_{in}(v), \forall v$. So $|S| = \sum_{v \in V} d_{in}(v)$.

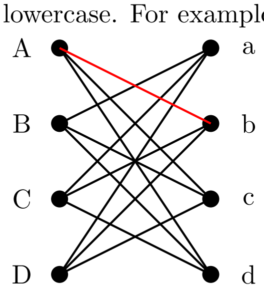
Therefore we conclude, $\sum_u d_{out}(u) = |E| = \sum_v d_{in}(v) = |S|$.

Q.2

(a) We claim that for any vertex v, $\deg(v) \geq 3$. Indeed, suppose $\exists v$ s.t. $\deg(v) < 3$, there're at most 2 edges incident to v. Removing such two edges, the resulting graph has an isolated vertex v, which is not connected, contradicting to the hypothesis that G is 3-connected.

Now by handshaking lemma, $2|E| = \sum_v d_v \geq \sum_v 3 = 3n$. Hence $|E| \geq \frac{3n}{2}$.

(b) Consider the following graph G, its vertices are labeled as $\{A, B, C, D, a, b, c, d\}$ where each capital letter is adjacent to the lowercase letters other than its own lowercase. For example, A is adjacent to b, c and d.



Now we remove two edges e_1, e_2 from G, if e_1 and e_2 share a same vertex, WLOG we can assume $e_1 = \{A, b\}$ the red edge. After we remove e_1, we obtain a graph with a cycle inside, namely

$$A, c, B, a, D, b, C, d, A.$$

As this cycle connects every vertex. After removing e_2 we still get a connected graph.
Other answers may apply to this question if appropriate.

Q.3
(a) Suppose, on the contrary, that $|d(v,u) - d(v,w)| > 1$. WLOG we assume $d(v,u) - d(v,w) > 1$. By definition we have a walk from v to w of length $d(v,w)$, say its points are recorded as $v_0 = v, v_1, \dots, v_k = w$ where $k = d(v,w)$. As w and u are adjacent, we have a walk from v to u of length $k+1$, namely

$$v_0 = v, v_1 \dots, v_k = w, v_{k+1} = u.$$

Hence $d(v,u) \leq d(v,w) + 1$, contradicting our assumption.
(b) Consider the set $V_i := \{u; d(v,u) = i\}$. Clearly $V_i \cap V_j = \emptyset$ for $i \neq j$. Also as G is connected, we can reach any vertices from v, $\cup_{i=0}^{\infty} V_i = V$ the whole set of vertices. To show that $\forall w \in V$, $d(v,w) \leq 3\lceil \frac{|V|}{\delta+1} \rceil$, it suffices to show $\forall k > 3\frac{|V|}{\delta+1}$, $V_k = \emptyset$. (Notice that $3\frac{|V|}{\delta+1} \leq 3\lceil \frac{|V|}{\delta+1} \rceil$.)
Next, we prove the following claim: if $|V_i| \neq 0$, then $|V_{i-1}| + |V_i| + |V_{i+1}| \geq \delta + 1$. Indeed, for $V_i \neq \emptyset$, we can choose a vertex $w_0 \in V_i$. By definition of minimum degree, we can choose δ vertices $w_1, \dots, w_\delta$ from the neighborhood of w_0. By Q.3.(a), we see that $|d(v,w_0) - d(v,w_j)| \leq 1$ for $j = 1, \dots, \delta$, which means $w_j \in V_{i-1} \cup V_i \cup V_{i+1}$. Hence $\{w_0, w_1 \dots, w_\delta\} \subset V_{i-1} \cup V_i \cup V_{i+1}$, proving our claim.
Now let N be the number of $i \geq 1$ s.t. $|V_i| \neq 0$, we have

$$\begin{aligned} 3|V| = 3\sum_{i=0}^{\infty} |V_i| &> \sum_{i=1}^{\infty} |V_{i-1}| + |V_i| + |V_{i+1}| \\ &\geq \sum_{i \geq 1, |V_i| \neq 0} |V_{i-1}| + |V_i| + |V_{i+1}|, \quad \text{summing over the } i\text{'s that } |V_i| \neq 0 \\ &\geq N(\delta+1), \quad \text{by previous paragraph.} \end{aligned}$$

Hence $N < \frac{3|V|}{\delta+1}$.
Notice that whenever $V_i \neq \emptyset$, we have $V_{i-1} \neq \emptyset$. Indeed if $v_0 = v, v_1, \dots, v_i \in V_i$ is a path from v to some vertex in V_i, we have $v_{i-1} \in V_{i-1}$.
Therefore the i's s.t. $|V_i| \neq 0$ are exactly the first N natural numbers. Hence $\forall k > \frac{3|V|}{\delta+1} > N$ we have $|V_k| = 0$, as wanted.