Math 154 Homework 5

Spring 2020

This homework is due on gradescope by Sunday May 17th at 11:59pm pacific time. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in LaTeXis recommend though not required.

Please cite any other students with whom you collaborated on any problems.

Question 1 (Semi-Regular Solids, 50 points)**.** *A semi-regular solid is a polyhedron where all faces are regular polygons and where at each vertex you have the same number of faces with the same numbers of sides. For example, in a pentagonal prism, each vertex is part of two square faces and one pentagonal face.*

(a) Show that if each vertex has a face with n_1 sides, a face with n_2 sides and so on through n_k then it must be the case that $v(1 - k/2 + 1/n_1 + 1/n_2 + \ldots + 1/n_k) = 2$. [25 points]

(b) Find all sequences of $n_1, n_2, \ldots, n_k$ for which there is such a semi-regular solid. [25 points]

Question 2 (Crossing Numbers, 50 points)**.** *The* crossing number *of a graph G (denoted $\mathrm{cr}(G)$) is the minimum number of pairs of edges that cross in any plane embedding of G. In particular, G is a planar graph if and only if $\mathrm{cr}(G) = 0$.*

(a) Show that if a graph G has e edges and v vertices that $\mathrm{cr}(G) \geq e - 3v + 6$. [20 points]
Note: a more sophisticated argument can be used to show that if $e \geq 4v$ that $\mathrm{cr}(G) \geq e^3/(64v^2)$. This can be shown by randomly throwing out all but a p-fraction of the vertices. This leaves only $p^4\mathrm{cr}(G)$ crossings (on average), but this still must be at least $p^2e - 3pv$. Taking $p = 4v/e$ gives the result.

(b) Show that if G has connected components $G_1, G_2, \ldots, G_m$ that $\mathrm{cr}(G) = \mathrm{cr}(G_1) + \mathrm{cr}(G_2) + \ldots + \mathrm{cr}(G_m)$. [10 points]

(c) Show that for any m, n with $m \leq \binom{n}{2}$ there is a graph G with m edges and n vertices with $\mathrm{cr}(G) \leq m^3/n^2$. [20 points]

Question 3 (Extra credit, 1 point)**.** *Approximately how much time did you spend on this homework?*