Math 154 Homework 1

Spring 2020

This homework is due on gradescope by Sunday April 12th at 11:59pm pacific time. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in LaTeXis recommend though not required.

Please cite any other students with whom you collaborated on any problems.

Question 1 (Handshake Lemma Generalizations, 40 points)**.** *The Handshake Lemma as stated in class applies only to simple, undirected graphs. With some effort, it can also be made to work for more general types of graphs, but one needs to be careful about how one defines the degree of a graph.*

(a) *For mutligraphs, we can no longer define the degree of a vertex by the number of neighboring vertices. Given a graph with vertices v and w connected by two edges, the degrees would then both be 1, but the total number of edges would be 2 and $1 + 1 \neq 2 \cdot 2$. Come up with a new definition of the degree of a vertex for multigraphs and prove the Handshake Lemma using this definition. [15 points]*
Hint: Remember that the original Handshake Lemma could be proved by counting the number of vertex-edge incidence pairs. How can you make this argument continue to work.

(b) *For pseudographs, we will again need to change the definition of degree in order to handle self loops. Come up with an appropriate definition of the degree of a vertex and prove the Handshake Lemma for pseudographs using this definition. [10 points]*

(c) *For directed graphs things are more complicated. One could ignore the direction of the edges and define the degree as before. Things will work, but if you are working with a directed graph it is probably because the directions of the edges matter. Thus, a more standard way to define things is to let the* in-degree *of a vertex v (denoted $d_{in}(v)$) be the number of edges pointing towards v, and the* out-degree *(denoted $d_{out}(v)$) be the number of edges pointing away from v. Using this notation prove that for any directed graph $G = (V, E)$ that*
$$\sum_{v \in V} d_{in}(v) = |E| = \sum_{v \in V} d_{out}(v).$$
This is known as the Handshake di-lemma. [15 points]

Question 2 (3-connected graphs, 20 points)**.** *Call a graph G k-connected if it remains connected even after removing any $k - 1$ edges of G. So for example, a 1-connected graph is the same as a connected graph, but a 2-connected graph cannot contain any* bridges*, that is any edges whose removal would disconnect the graph.*

(a) *Show that any 3-connected graph with n vertices must have a total of at least $3n/2$ edges. [10 points]*

(b) *Find an example of a 3-connected graph with 8 vertices and 12 edges. [10 points]*

Question 3 (Distances and Minimum Degrees, 40 points)**.** *In a connected graph G, define the* distance *between two vertices v and w (denoted $d(v, w)$) be the length of the shortest walk between v and w.*

(a) *Show that for any vertices u, v, w with v and w adjacent that $d(u, v)$ and $d(u, w)$ differ by at most 1. [15 points]*

(b) *Let $\delta(G)$ denote the minimum degree of any vertex in G. Show that if $G = (V, E)$ is connected, then for any two vertices v, w that $d(v, w) \leq 3 \left\lceil \frac{|V|}{\delta(G)+1} \right\rceil$. [25 points]*
Hint: Sort the vertices of G based on their distance from v. Use part (a) to show that for any i there must be many vertices at distance $i - 1, i,$ or $i + 1$.

Question 4 (Extra credit, 1 point). *Approximately how much time did you spend on this homework?*