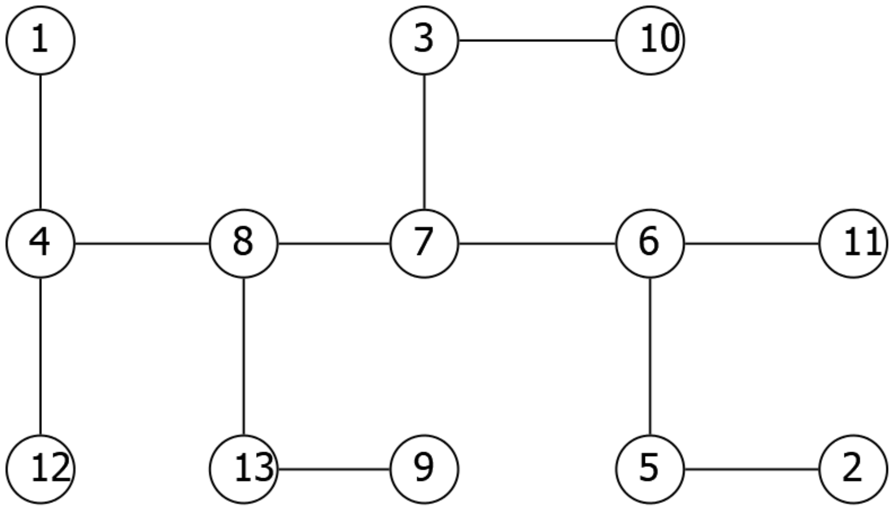
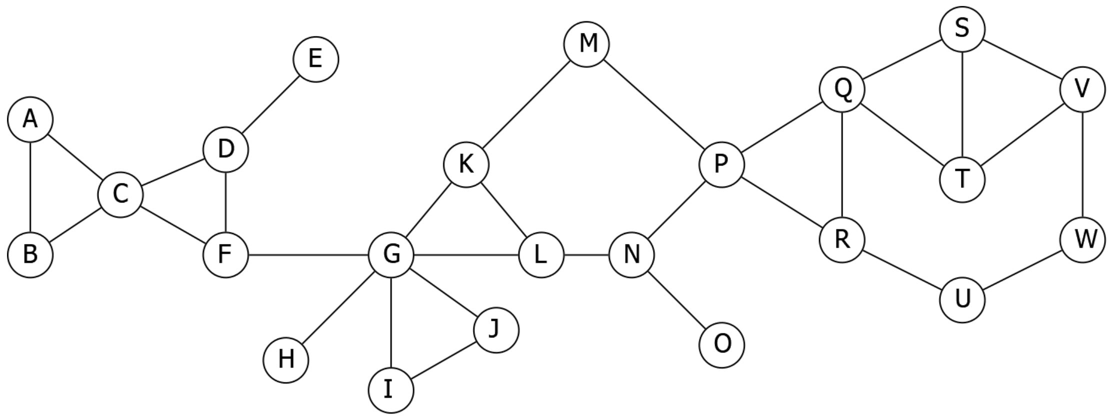
Question 1 (Tree List, 25 points)**.** *Our proof of Caley's Theorem described how to associate a list of numbers to a labelled tree. What is the list corresponding to the tree given below?*



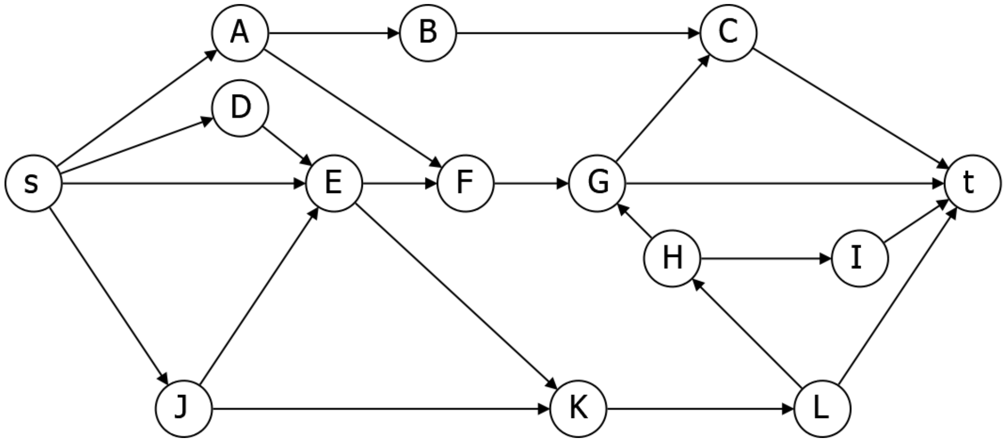
We repeatedly remove the leaf with the smallest index and record the index of its neighbor. This gives the sequence: 4,5,6,13,3,7,6,7,8,4,8.

Question 2 (Block Decomposition, 25 points)**.** *What are the blocks in the graph below?*

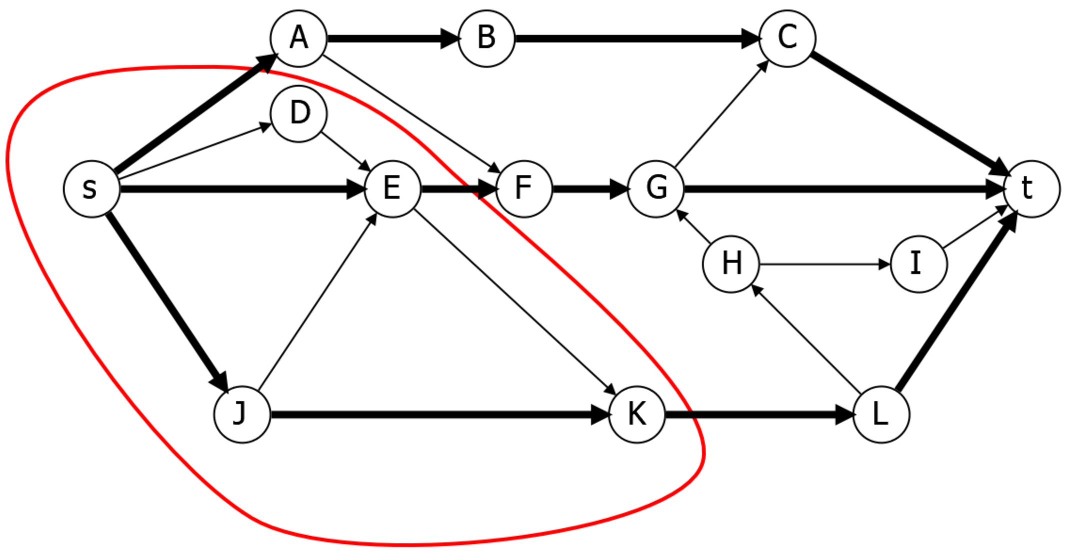


The cut vertices are C, D, F, G, N, P. Looking at the sets of vertices between the cuts, we get blocks: ABC, CDF, DE, FG, GH, GIJ, GKLMNP, NO, PQRSTUVW.

Question 3 (Minimum Cut, 25 points)**.** *Give a minimum cut for the following network.*

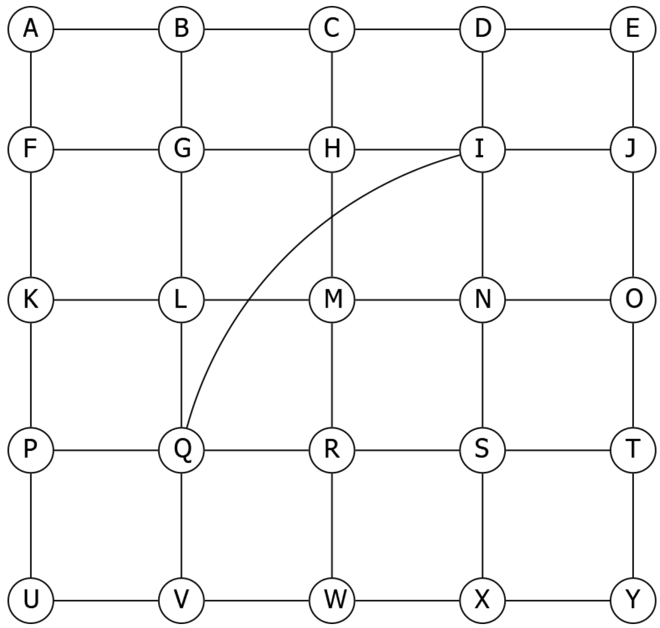


We compute a maximum flow, for example the one given below: We note that the specified cut matches

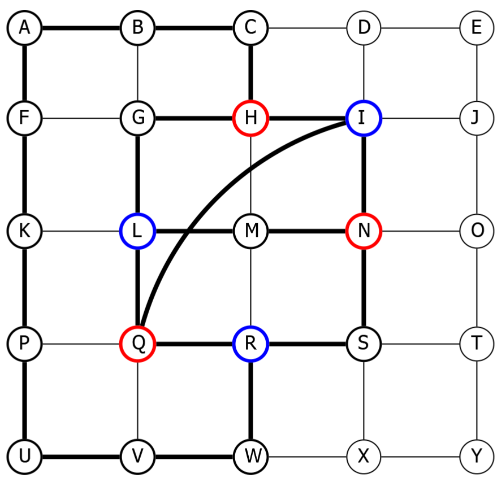


the flow and must therefore be minimal. In particular, our cut has s,D,E,J,K on one side and all other vertices on the other.

Question 4 (Planarity, 25 points)**.** *Prove that the graph below is not planar.*



We note that the following is a subgraph that is a subdivision of a $K_{3,3}$ with the parts of the $K_{3,3}$ being the red and blue vertices, respectively. Therefore, it is not planar.



Question 5 (Sum of Eulerian Graphs, 25 points)**.** *Suppose that G and H are two connected, Eulerian graphs on the same vertex set that do not share any edges. Show that $G \cup H$ is Eulerian where $G \cup H$ denotes the graph whose edges are all of those in either G or H.*

It is enough to show that $G \cup H$ is connected and has every vertex of even degree. To show that it is connected, we note that by assumption since G is connected, there is a path between any pair of vertices in G. Since these paths also exist in $G \cup H$, there are paths between every pair of vertices in that graph, and therefore, it is also connected. Next, we need to show that every vertex in $G \cup H$ has even degree. However, it is easy to see that for any vertex v that $d_{G \cup H}(v) = d_G(v) + d_H(v)$. Since G and H are Eulerian, we must have $d_G(v)$ and $d_H(v)$ are both even. Since the sum of two even numbers is even, $d_{G \cup H}(v)$ is even for all v. Therefore, $G \cup H$ is Eulerian.

Question 6 (Chromatic Number of Unions, 25 points)**.** *Let G and H be two graphs sharing the same vertex set. Let $G \cup H$ denote the graph consisting of the union of their edges. Show that $\chi(G \cup H) \leq \chi(G) + \tau(H)$ where $\tau(H)$ denotes the size of the minimum vertex cover of H.*

Let S be a minimum vertex cover of H. So $|S| = \tau(H)$. Begin by coloring G using $\chi(G)$ colors. Then assign each element of S a new, distinct color. This is clearly a way of coloring the vertices of $G \cup H$ using at most $\chi(G) + \tau(H)$ distinct colors. We need to show that it is a valid coloring, namely that and two adjacent vertices, u and v are colored differently.

If either u or v is in S, then it is colored a unique color and thus, the two are different colors. Otherwise, since S is a vertex cover of H, the edge u, v cannot be in H (since every edge in H will have a vertex in S). Thus, since the edge is in $G \cup H$, (u, v) is an edge in G. Since our initial coloring of G is valid, u and v must have been assigned different colors by this coloring, which are still the colors that they have after recoloring the vertices of S.

This completes our proof.

Question 7 (Perfect Matchings and Degrees of Independent Triples, 25 points). *Let G be a connected graph on $2n$ vertices. Suppose that for any vertices u, v, w of G with no two of them connected by an edge, $d(u) + d(v) + d(w) \geq 3n - 2$. Show that G has a perfect matching.*

Assume for sake of contradiction that G does not have a perfect matching. We note by Tutte's Theorem that there must be a set S with $|S| < \Omega(G - S)$. Since G is connected and has an even number of vertices, $|S|$ must be at least 1, and by the parity argument $\Omega(G-S)$ must be at least 3. Let u, v and w be from the three smallest components of $G - S$ (there must be at least 3 since $\Omega(G - S)$ is at least 3). We note that u, v, and w can only have neighbors either in S or as other elements of their component. If $|S| = m < n$, then the contribution of neighbors in S to the sum of their degrees is at most $3m$. The sum of the sizes of their components is at most $\lfloor 3(2n - m)/(m + 2) \rfloor$, since there are at least $m + 2$ components using up the remaining $2n - m$ vertices and we picked the smallest 3. Therefore we have that

$$d(u) + d(v) + d(w) \leq 3m + \lfloor 3(2n - m)/(m + 2) \rfloor - 3.$$

We claim that this is at most $3n-3$, which would give us a contradiction. If not, $3m+3(2n-m)/(m+2) \geq 3n$. However, it is easy to see that $3m+3(2n-m)/(m+2)$ is maximized when m takes one of the extreme values $m = 1$ or $m = n - 1$, but in either case the inequality fails. This completes our proof.

Question 8 (Cycle and Star Ramsey Numbers, 25 points). *Show for any positive integer n that* $R(C_{2n}, K_{1,n}) = 2n$.
Hint: There is a result from early in the course that will be useful here.

We note that $R(C_{2n}, K_{1,n}) \geq 2n$ since if you color the edges of a K_{2n-1} all red, you have no red C_{2n} nor blue $K_{1,n}$.

To show the other direction consider a red-blue coloring of the edges of a K_{2n}. Let G be the subgraph consisting of the red edges. If $\delta(G) < n$ then there is some vertex v with at least $2n - 1 - (n - 1) = n$ blue edges out of it. This gives us a blue $K_{1,n}$. Otherwise, we have a graph on $2n$ vertices with minimum degree at least $n = (2n)/2$. Therefore, as proved in class, G has a Hamiltonian cycle. This cycle correspond to a red C_{2n} in our original graph. Thus, any coloring has either a red C_{2n} or a blue $K_{1,n}$ and this completes our proof.