

Math 154 Homework Solution

Fall 2021

Solution to Homework 6

This homework is due on gradescope by Friday November 12th at 11:59pm pacific time. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in L^AT_EX is recommended though not required.

Please cite any other students with whom you collaborated on any problems.

Question 1 (Uniqueness of Planar Embeddings, 40 points). *(a) Let $G = (V, E)$ be a connected, planar graph with $|E| = 3|V| - 6$. Show that any two planar embeddings of G have the same set of faces (in particular, that if one embedding has a face whose sides consist of some collection of edges, then the other planar embedding will have a face with the same sides). Hint: Note that G must be triangulated. Consider the faces including a given vertex v . [30 points]*

(b) Show that this is no longer the case if we drop the condition $|E| = 3|V| - 6$. In particular, give a connected, planar graph G with two different embeddings that have different faces. [10 points]

Proof. By Euler's formula, $|V| = 2 + |E| - |F|$. Thus, $|E| = 3|V| - 6 = 3|E| - 3|F|$, and equivalently, $2|E| = 3|F|$. Since $|E| > 0$, $|V| \geq 3$, and thus every face has at least 3 sides. By the dual hand shake lemma, $3|F| = 2|E| = \sum_{f \in F} \text{sides}(f) \geq 3|F|$. Thus, the $\geq$ in the equation is actually an equality, which is true if and only if every face has exactly 3 sides. Hence, G is triangulated.

Let $A, B : G \rightarrow \mathbb{R}^2$ be two embeddings of G , and f be a face in A . Then, f has 3 sides, say $e_1 = \{u, v\}$, $e_2 = \{v, w\}$ and $e_3 = \{w, u\}$. It suffices to show that in the embedding B , e_1 , e_2 and e_3 are the sides of a face. Let D be the closed region bounded by edges e_1 , e_2 and e_3 in the embedding B .

Case 1: If D contains no other vertices, then e_1 , e_2 and e_3 form a face.

Case 2: Suppose that D contains some vertex x other than u, v , and w . Suppose for contradiction that there exists a vertex y such that $B(y) \notin D$. Then, since G is connected, there exists a path P from y to x . The first intersection point of $B(P)$ and D must be among $\{u, v, w\}$ since B is an embedding of planar graph. By relabelling the vertices, we may assume without loss of generality that the first intersection point is v . Then, along the path P , there is a neighbour v_1 of v that is outside of D and a neighbour v_2 of v that is inside of D . Consider the neighbours of v in the embedding A . Since every face has only 3 sides, by tracing the edges between the neighbours of v , we have a cycle containing all neighbours of v , on which u and w are neighbours. Thus, there is a path P' from v_2 to v_1 which does contain u, v or w . However, by the same argument above, any path from v_2 (outside D) to v_1 (inside D) has to pass through u, v , or w , contradiction. Hence, there is no vertex y with $B(y) \notin D$. Thus, D contains all vertices, and e_1 , e_2 and e_3 form a face. $\square$

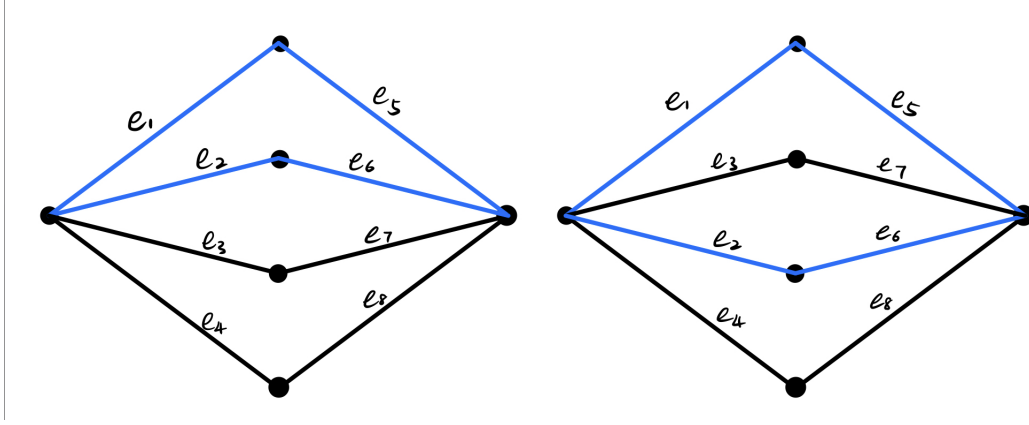


Figure 1: The graph G and two embeddings

Proof. In the first embedding, there is a face with sides $\{e_1, e_2, e_5, e_6\}$, while in the second embedding, there is no face with sides $\{e_1, e_2, e_5, e_6\}$. $\square$

Question 2 (Triangle-less Chromatic Number, 30 points). *Give an example of a graph G which contains no triangle and which has chromatic number at least 4.*

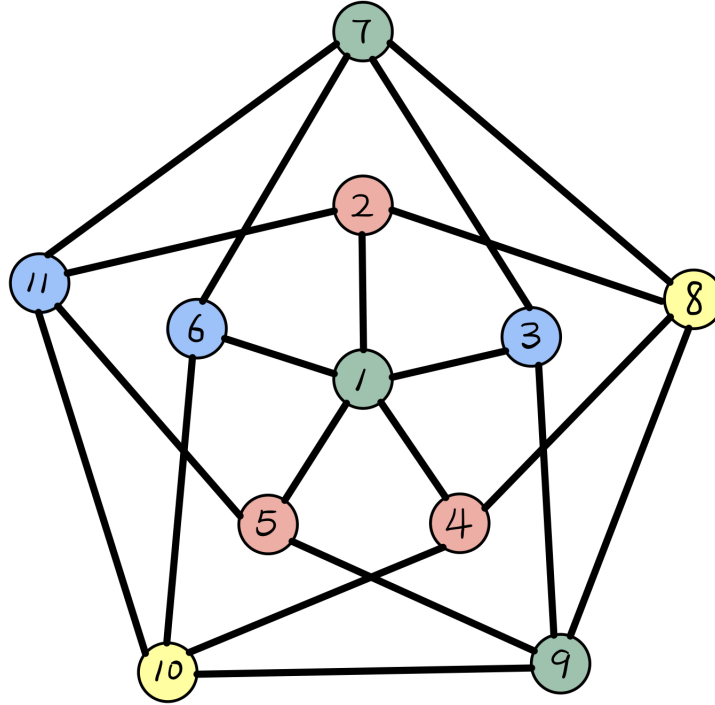


Figure 2: The graph G and a 4-coloring of G

Proof. Let G be the graph as in figure 2. It has no triangles. Suppose for contradiction that G is 3-colorable by colors $\{a, b, c\}$. Suppose without loss of generality that vertex 1 is of color a . Since the outer cycle is odd, all three color will be used on the outer cycle, in particular, there is one vertex of color a . Suppose without loss of generality that vertex 7 has color a , and that vertex 2 is of color b . Then since vertex 2 has color b

and vertex 7 has color a , vertex 8 and 11 both have color c . Then, based on the colors on vertex 1, 8, and 11, vertex 5 and 4 should have color b . Thus, vertex 9 and 10 are both of color a , which is a contradiction to the fact that they are neighbours. $\square$

Question 3 (Number of Colorings for Graphs with Small Maximum Degree, 30 points). *Let G be a connected graph with m vertices and let $n \geq \Delta(G) + k$ for some positive integer k . Show that there are at least $(k+1)^{m-1}$ ways to color G with any set of n colors.*

Proof. Run the BFS algorithm and let $(v_1, v_2, \dots, v_m)$ be the sequence of all vertices in the order we discover them. Then for $i \geq 2$, v_i has a neighbour v_j for $j < i$. Thus, for $i \geq 2$, the number of neighbours of v_i after itself is most $\Delta(G) - 1$.

Color the vertices using the greedy algorithm in the order $(v_m, \dots, v_2, v_1)$. That is, choose an arbitrary color for v_m , and given the coloring of $(v_m, \dots, v_{i+1})$, choose an arbitrary color for v_i that is different from the colors of its neighbours that are already colored. For each edge $\{v_i, v_j\}$, suppose $i < j$, and then by the way we choose the color on v_i , its color is different from that of v_j . Then, for each $i \geq 2$, at the step where we choose the color of v_i , v_i has at most $\Delta(G) - 1$ colored neighbours, and we have at least $n - (\Delta(G) - 1) = k + 1$ choices for the color of v_i . Thus, this procedure gives at least $(k + 1)^{m-1}$ distinct colorings of G . $\square$

Question 4 (Extra credit, 1 point). *Free point!*