

Math 154 Homework Solution

Fall 2021

Solution to Homework 5 By John Li

This homework is due on gradescope by Friday November 5th at 11:59pm pacific time. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in L^AT_EX is recommended though not required.

Please cite any other students with whom you collaborated on any problems.

Question 1 (Block Structure of Trees, 20 points). *Let T be a finite tree. Show that T is the block graph of some other tree T' if and only if the following condition holds: T is a bipartite graph where all of the leaves are in the same part and the vertices in that part all have degree 1 or 2.*

" $\implies$ "

Assume that T is the block graph of tree T' . By definition, T has some vertices that correspond to blocks in T' and the other vertices correspond to cut vertices in T' , and all edges in T connect one to the other. So we can identify T as the bipartite graph where one part is all the vertices correspond to blocks and the other part is all the vertices correspond to cut vertices (See Figure 1). The leaves in T correspond to blocks, since a cut vertex would separate blocks. Hence all of the leaves are in the same part.

Now we prove that all the vertices correspond to blocks have degree at most 2. Since T' is a tree, each block is a K_2 , either a leaf and its neighbor, or two adjacent non-leaf vertices. Hence in T the vertices correspond to blocks can either be a leaf or be connected to 2 vertices correspond to cut vertices. So the degree is at most 2.

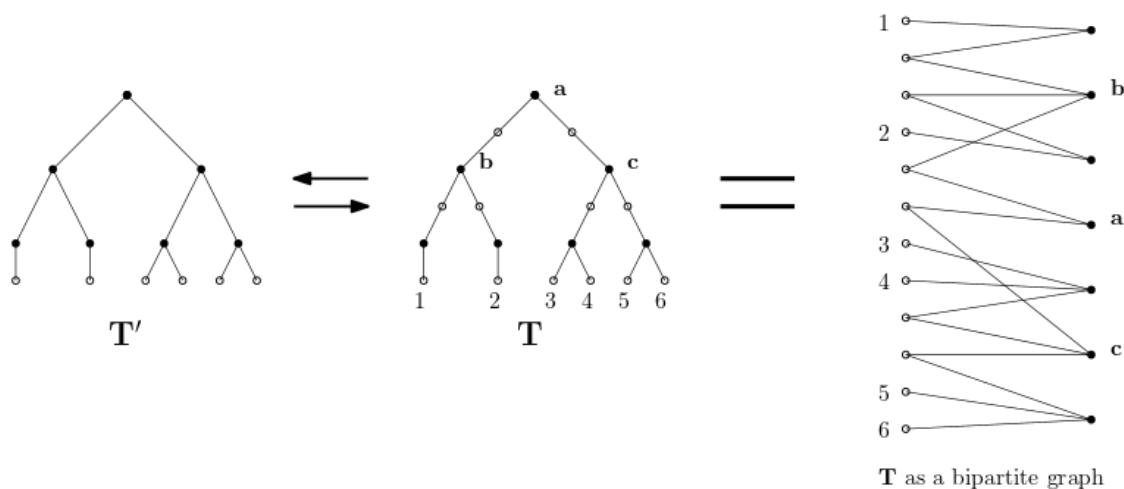


Figure 1: An example of a tree T' and its block tree T .

" $\Leftarrow$ "

Assume that T is a bipartite graph where all of the leaves are in the same part and the vertices in that part all have degree 1 or 2. We can construct T' in the following way:

The vertices of T' are the leaves of T and the vertices of T in the other part of the bipartition.

The edges of T' connect a leaf to its adjacent vertex or two cut vertices that have a single block vertex between them.

Inserting a block vertex between any pairs of adjacent cut vertices of T' , we get T , hence T is the block tree of T' .

Question 2 (Distinct Paths in a Block, 40 points). *Let G be a finite graph that consists of a single block with more than 2 vertices. Let v and w be two different vertices of G .*

(a) *Show that for any two distinct vertices s and t it is either possible to find an $s - v$ path and a $t - w$ path that do not share any vertices or it is possible to find a $t - v$ path and an $s - w$ path that do not share any vertices. [20 points]*

(b) *Show that if G has m edges and n vertices that there are at least $m - n + 2$ different paths from v to w . Hint: Use induction on the number of ears in an ear decomposition of G . [20 points]*

(a) By theorem 4.2.4 proven in class, vertex v and s share a cycle C . Let p be the $t - w$ path. If p doesn't intersect with C , then we can take an $s - v$ path in C and it doesn't share any vertices with p , we are done.

If p intersects with C , let x be the vertex where p intersects C at the first time, and y be the vertex where p intersects C at the last time.

- Case 1, if x and y are on the same side of C , then there is an $x - y$ path p' on C that doesn't contain s or v . We can connect s and v by the other side of cycle, and connect t and w by the $t - x - p' - y - w$ path. By construction these two paths doesn't share vertices.
- Case 2, if x and y are on different sides of C , then on C there is an $s - y$ path and an $x - v$ path such that they don't share vertices. We can connect s and w by the $s - y - w$ path and connect t and v by the $t - x - v$ path.
- Case 3, x or y coincide with s or v , i.e., p passes through s or v . By theorem 4.2.4, t and w share a cycle, so there is another $t - w$ path p' which doesn't share vertices with p except t and w . If p passes through both s and v , then p' cannot pass through either s or v , so we go back to previous cases. If p passes through s , and p' passes through v , then p contains an $s - t$ path, and p' contains a $v - w$ path, by construction they don't share vertices.

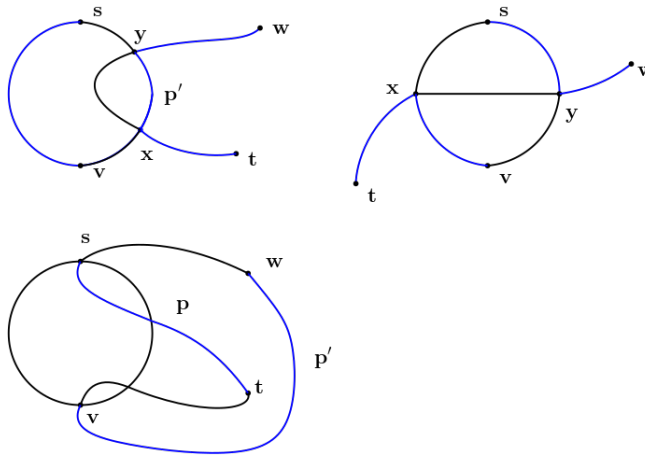


Figure 2: Constructing $s - v$ and $t - w$ paths in different cases.

- (b) Note that G is a cycle plus ears, we can do induction on the number of ears k .

When $k = 0$, G is a cycle, $m - n + 2 = 2$, and there are two $v - w$ paths.

When G has $k - 1$ ears, since by adding one ear we add a path plus two edges, we increase $m - n$ by 1, so $m - n + 2 = k - 1 + 2 = k + 1$. As inductive hypothesis, we assume that there are at least $k + 1$ paths that connect v and w . When G has k ears, $m - n + 2 = k + 2$. Let s, t be the start and end vertices of the ear P which doesn't contain w or v . From part (a) we know that we have an $s - v$ path and a $t - w$ path that do not share any vertices, or we have a $t - v$ path and an $s - w$ path that do not share vertices. Hence we can define a new $v - w$ path as $v - s - P - t - w$ or $v - t - P - s - w$ which is different from the $k + 1$ paths that connect v and w in $G - P$, since only the new path contains P . So we have $k + 2$ different $v - w$ paths in G . By induction, we have shown that G has $m - n + 2$ different $v - w$ paths.

Question 3 (Euler's Formula and Regular Tessellations, 40 points). *Let G be an infinite graph with a planar embedding. Say that the embedding is periodic if translating all of the edges and vertices one unit up or to the left yields the same collection of edges and vertices. Say that the embedding is locally finite if any disk intersects only finitely many edges and vertices.*

- (a) *For G a locally finite, periodic planar graph, say that two vertices are equivalent if you can obtain one from the other by translating it an integer number of units up and an integer number of units to the left. Let v be the number of equivalence classes of vertices. Similarly, let e and f be the number of equivalence classes of edges and faces of G . Prove that $v - e + f = 0$. [20 points]*
- (b) *Suppose that such a graph G is d -regular and has exactly s sides per face. What are all possible values of d and s for which this can happen? [20 points]*
- (a) Since G is locally finite, the intersection of G and a disk is a finite subgraph H . Let V, E, F be the number of vertices, edges and faces of H , since H has a planar embedding, by Euler formula we have

$$V - E + F = 2. \quad (1)$$

To relate V, E, F with v, e, f , we assume H contains N_1 units, defined as the minimal subgraph that generates G by translation (see Figure 3¹) and intersects with N_2 units. Obviously $N_1 \leq N_2$.

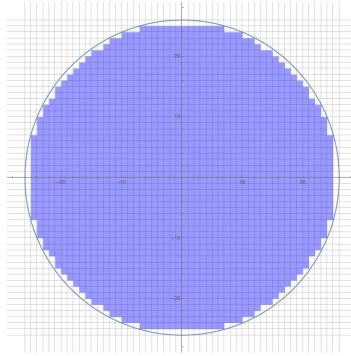


Figure 3: the shape in blue is the N_1 units contained in the disk.

Since H has more edges, vertices and the infinite face than N_1 units, we have

$$V \geq N_1 v, E \geq N_1 e, F \geq N_1 f, \quad (2)$$

on the other hand, H is contained in N_2 units, we have

$$V \leq N_2 v, E \leq N_2 e, F \leq N_2 f, \quad (3)$$

¹Source: <http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Numbers/year2016.html>

hence

$$2 = V - E + F \geq N_1 v - N_2 e + N_1 f, \quad (4)$$

rearranging, we have

$$v - \frac{N_2}{N_1} e + f \leq \frac{2}{N_1} \quad (5)$$

Note that this is true for any size of disk. If we let the radius of the disk go to infinity, then N_1 and N_2 will go to infinity, but $\frac{N_2}{N_1}$ will go to 1, since for any finite graph, when the disk is large enough to cover the entire graph, we have $N_1 = N_2$. Hence we have

$$v - e + f \leq 0. \quad (6)$$

Meanwhile,

$$2 = V - E + F \leq N_2 v - N_1 e + N_2 f, \quad (7)$$

rearranging, we have

$$v - \frac{N_1}{N_2} e + f \geq \frac{2}{N_2}, \quad (8)$$

let the radius of the disk go to infinity, using the same argument, we have

$$v - e + f \geq 0, \quad (9)$$

hence we have

$$v - e + f = 0. \quad (10)$$

- (b) Using the same technique as (a), we consider the finite subgraph H which is the intersection of G and a disk. By applying the handshake lemma and the dual handshake lemma to H , we have

$$2E = dV, \quad 2E = sF, \quad (11)$$

and from (2)(3) we have

$$2N_1 e \leq 2E = dV \leq dN_2 v, \quad (12)$$

hence

$$2e \leq \frac{N_2}{N_1} dv, \quad (13)$$

let the radius of the disk go to infinity, in (a) we have shown that $\frac{N_2}{N_1}$ will go to 1, so we have

$$2e \leq dv. \quad (14)$$

Meanwhile

$$2N_2 e \geq 2E = dV \geq dN_1 v, \quad (15)$$

hence

$$2e \geq \frac{N_1}{N_2} dv, \quad (16)$$

let the radius of the disk go to infinity we have

$$2e \geq dv, \quad (17)$$

so from (14)(17) we have

$$2e = dv. \quad (18)$$

Similarly, from the dual handshake lemma in (11), we have

$$\frac{N_1}{N_2}sf \leq 2e \leq \frac{N_2}{N_1}sf, \quad (19)$$

let the radius of the disk go to infinity we have

$$2e = sf. \quad (20)$$

Plug equation (18)(20) in (10) we have

$$\frac{2e}{d} - e + \frac{2e}{f} = 0, \quad (21)$$

rearranging, since $e \neq 0$, d, s are the solution of

$$\frac{1}{d} + \frac{1}{s} = \frac{1}{2}, \quad (22)$$

and we have

$$d = \frac{2s}{s-2} = 2 + \frac{4}{s-2}. \quad (23)$$

Since d, s are positive integers, $s-2$ must divide 4, hence s can only be 3, 4 or 6. So all possible values of d and s are $\{d=3, s=6\}$, $\{d=4, s=4\}$ and $\{d=6, s=3\}$. See Figure 4.

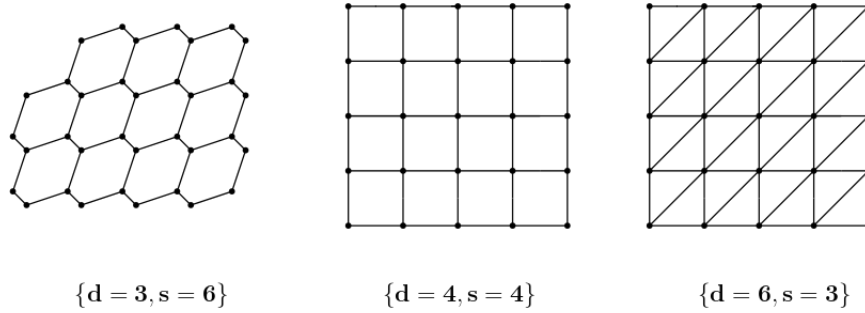


Figure 4: The regular tessellations.

Question 4 (Extra credit, 1 point). *Approximately how much time did you spend on this homework?*