

Math 154 Homework Solution

Fall 2021

Solution to Homework 4 By Haixiao Wang

This homework is due on gradescope by Friday October 22nd at 11:59pm pacific time. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in L^AT_EX is recommended though not required.

Please cite any other students with whom you collaborated on any problems.

Question 1 (Infinite Semi-Eulerian Graphs, 80 points). *Define a semi-infinite trail on a graph G to be a sequence of vertices $v_1, v_2, v_3, \dots$ so that for each i there is an edge in G between v_i and v_{i+1} and so that these edges are all distinct. Call G semi-Eulerian if there exists such a trail that uses every edge of G exactly once. For this problem, we will consider graphs G with infinitely many vertices and edges for which each vertex is incident on only finitely many edges (and thus has a well defined degree).*

- (a) *Show that if G has a semi-infinite Eulerian trail starting at a vertex v , then $\deg(v)$ is odd and all other degrees in G are even, and that G is connected except for isolated vertices. [20 points]*
- (b) *Unfortunately, the conditions in (a) are not sufficient. Give an example of a graph G with a vertex v of odd degree and all other degrees even so that G does not have a semi-Eulerian trail. [20 points]*
- (c) *Show that any graph G with a semi-infinite Eulerian trail must satisfy the following property: If G' is any subgraph of G obtained by removing finitely many edges of G , then G' contains only one connected component of infinite size. [20 points]*
- (d) *It turns out that the conditions in (a) and (c) together are sufficient. The proof relies on the following **Lemma**: Suppose G is a graph with only one vertex v of odd degree and satisfying the condition in (c). If P is any finite path starting at v , then the edges of all of the finite connected components of $G - P$ can be partitioned into circuits. Prove this lemma. [20 points]*

Proof. (a) Let P denote the semi-infinite trail starting at vertex v . Consider vertex $u \neq v$, which is not the starting point. When there is an edge in, there must be an edge out, since the trail would not stop at u and every edge at u is used here, then the out degree always equals in degree, hence

$$\deg_{out}(u) = \deg_{in}(u) \implies \deg(u) = \deg_{out}(u) + \deg_{in}(u) = 2 \deg_{in}(u). \quad (1)$$

However, for the starting point v , the argument above holds true except for the first step out, then difference between out degree and in degree should be 1, hence

$$\deg_{out}(v) = \deg_{in}(v) + 1 \implies \deg(v) = \deg_{out}(v) + \deg_{in}(v) = 2 \deg_{in}(v) + 1. \quad (2)$$

- (b) The counter example can be seen in Figure 1, where the trail starting at v can't be semi-infinite Eulerian, since a trail starting at v can explore only one branch and it would never come back to go through vertices in other branches.

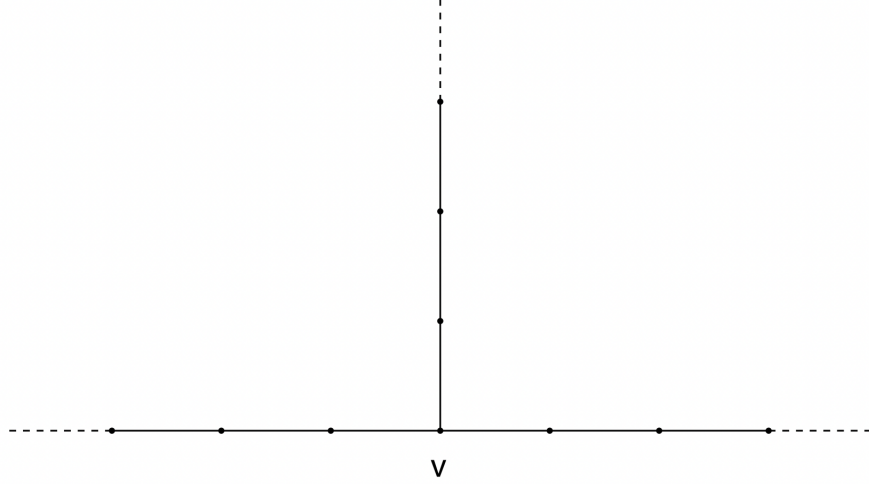


Figure 1: Counter example for part (b).

- (c) Let $\{f_1, f_2, \dots, f_k\}$ denote the set of finitely many edges removed from G , i.e., $E(G') = E(G) \setminus \{f_1, f_2, \dots, f_k\}$. We assume that there are 2 connected components of infinite size in G' .

Note that G contains a semi-infinite Eulerian trail $(e_1, e_2, e_3, \dots)$, which contains all edges in G , then each f_j is some e_{i_j} . Let $N = \max_{1 \leq j \leq k} i_j$ and consider the finite trail $P = (e_1, e_2, \dots, e_N)$. Define $G'' := G - P$, which contains a semi-infinite Eulerian trail $(e_{N+1}, e_{N+2}, \dots)$, thus G'' is connected except for isolated vertices.

On the other hand, G'' is a subgraph of G' , obtained by removing finitely many edges $\{e_1, e_2, \dots, e_N\} \setminus \{f_1, \dots, f_k\}$. However, by our assumption, G' contains 2 infinite components. Neither of them can be eliminated by removing finitely many edges. Then by this construction, G'' must have 2 disjoint connected components, indicating that G'' is disconnected, which contradicts to the conclusion above.

- (d) Let $P = (v, \dots, u)$ denote the finite path starting at v , with u being the endpoint. When P is removed from G , there is only one vertex in $G - P$ with odd degree, which turns out to be u due to conditions in (a). The connected component containing u , denoted by G' , should have infinite size, otherwise it would contradict to the *HandShake* Lemma since the number of vertices with odd degrees is odd. Let $H \subset (G - P)$ be a connected component of $G - P$, which is disjoint from G' . By discussion above, any vertex $w \in H$ should have even degree. Therefore H is Eulerian, which can be partitioned into circuits. An example can be seen in Figure 2.

□

Question 2 (Toughness of Hamiltonian Graphs, 20 points). Let G be a Hamiltonian graph and let $\{v_1, v_2, \dots, v_k\}$ be a set of k vertices of G for some $k \geq 1$. Let G' be the subgraph of G obtained by removing all of the vertices v_i and all of their edges from G . Prove that G' has at most k connected components.

Proof. Let C be the Hamiltonian cycle in graph $G = (V, E)$ with $|V| = n$, then $V(C) = V(G) = n$. The cycle C with k vertices removed is always a disjoint union of at most k paths, which can be proved by following induction.

1. $k = 1$: By removing vertex v_1 and its incident edges in C , we would obtain a path $P = (v_2, \dots, v_n)$.
2. $k = 2$: If we remove vertex v_i with $3 \leq i \leq n - 1$, then we get 2 connected components $(v_2, \dots, v_{i-1})$ and $(v_{i+1}, \dots, v_n)$; otherwise we would obtain only one connected component, either $(v_3, \dots, v_n)$ or $(v_2, \dots, v_{n-1})$.
- ...

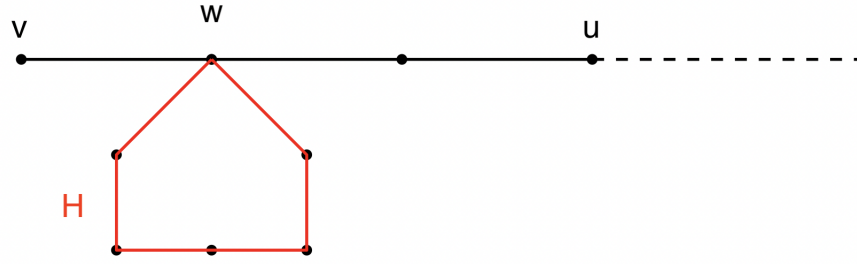


Figure 2: An illustrating example for part (d).

3. $k = m + 1$: Suppose that we can obtain at most m disjoint paths if m vertices are removed. When we are going to remove the $(m + 1)^{\text{th}}$ vertex and its incident edges, the number of connected components will not increase when we remove isolated vertex or ends of some path. The only way to increase the number is to remove a middle vertex in some path to break the path into 2 pieces, which will only increase the number by 1. Therefore we can obtain at most $m + 1$ disjoint paths when $m + 1$ vertices are removed, which finishes the induction argument.

Now, the cycle C is partitioned into at most k disjoint pieces $P_1, P_2, \dots, P_k$ when k vertices and their incident edges are removed, then $V(G') = V(P_1) \cup V(P_2) \cup \dots \cup V(P_k)$. The subgraph G' can be obtained from $P_1, P_2, \dots, P_k$ by adding edges outside the Hamiltonian cycle C , which will not produce new connected component. Therefore, G' has at most k connected components. $\square$

Question 3 (Extra credit, 1 point). *Approximately how much time did you spend on this homework?*