

Math 154 Homework Solution

Fall 2021

Solution to Homework 3 Qingyuan Chen

This homework is due on gradescope by Friday October 15th at 11:59pm pacific time. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in L^AT_EX is recommended though not required.

Please cite any other students with whom you collaborated on any problems.

Question 1 (Short Cycles, 30 points). *If G is a connected graph on n vertices with n edges, we know it must contain a cycle. Prove that if G has $n + 1$ edges, it must contain a cycle of length at most $(2n/3) + 2$. Hint: Consider G as a tree plus two extra edges. Consider how the tree connects those 4 endpoints and how to connect them up to make a cycle.*

Proof. Let G be a connected graph on n vertices with $n + 1$ edges. Assume for contradiction that all cycles in G has length larger than $(2n/3) + 2$. Let T be a spanning tree of G . Since T has $n - 1$ edges, there are two different edges $e_1, e_2 \in E(G) \setminus E(T)$. Since $T \cup \{e_1\}$ and $T \cup \{e_2\}$ are both graphs on n vertices with n edges, there are cycles C_1 and C_2 such that $e_1 \in E(C_1) \setminus E(C_2)$ and $e_2 \in E(C_2) \setminus E(C_1)$. Then,

$$|E(C_1) \cap E(C_2)| = |E(C_1)| + |E(C_2)| - |E(C_1) \cup E(C_2)| > 2(2n/3 + 2) - n = n/3 + 4,$$

and $E(C_1) \cap E(C_2) \subseteq E(T)$. Let P_1 be C_1 with e_1 removed and P_2 be C_2 with e_2 removed. Then, both P_1 and P_2 are subgraphs of T , and $E(P_1) \cap E(P_2) = E(C_1) \cap E(C_2) \neq \emptyset$. We will show that their intersection is a path in T .

Claim. *Let T be a tree and P, Q be paths in T . If P and Q share an edge, then their intersection $P \cap Q$ is a path in T .*

Proof. Suppose that $P = ((u_1, u_2, \dots, u_j))$. Let $a = \min\{i : \{u_i, u_{i+1}\} \in E(P) \cap E(Q)\}$ and $b = \max\{i : \{u_{i-1}, u_i\} \in E(P) \cap E(Q)\}$. Since $E(P) \cap E(Q) \neq \emptyset$, a and b are well defined, and $a < b$. Let R be the path $(u_a, u_{a+1}, \dots, u_b)$. Then, R contains $P \cap Q$. Since both P and Q contains a path from u_a to u_b and such path is unique in T , R is a subgraph of $P \cap Q$. Thus, $P \cap Q = R$ is a path in T . $\square$

By the claim above, the intersection of P_1 and P_2 is a path. Denote this path by P_3 , and its endpoints by s and t . Let P_4 be C_1 with edges $E(P_3)$ and vertices $V(P_3) \setminus \{s, t\}$ removed, and P_5 be C_2 with edges $E(P_3)$ and vertices $V(P_3) \setminus \{s, t\}$ removed. Then, both P_4 and P_5 are paths from s to t in G , and by construction, $E(P_4) \cap E(P_5) = \emptyset$. Let C_3 be the union of P_4 and P_5 in G . Then, C_3 is a cycle, and

$$\begin{aligned} |E(C_3)| &= |E(P_4)| + |E(P_5)| \\ &= |E(C_1)| + |E(C_2)| - 2|E(P_3)| \\ &= |E(C_1) \cup E(C_2)| + |E(C_1) \cap E(C_2)| - 2|E(P_3)| \\ &= |E(C_1) \cup E(C_2)| + |E(P_3)| - 2|E(P_3)| \\ &= |E(C_1) \cup E(C_2)| - |E(P_3)| \\ &< n - (n/3 + 4) \\ &= 2n/3 - 4, \end{aligned}$$

contradiction. $\square$

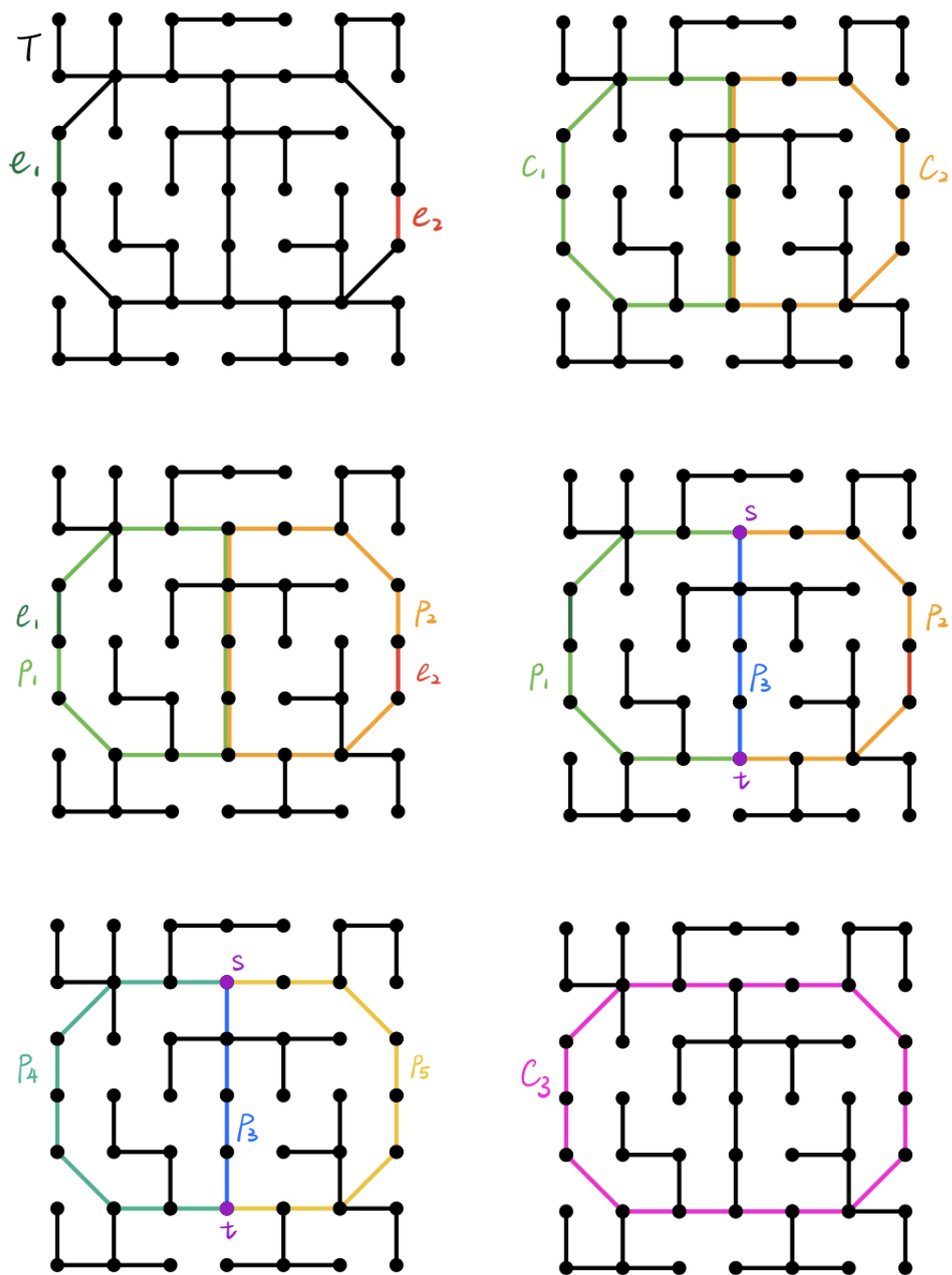


Figure 1: An example of the constructions in the proof

Question 2 (MSTs Have the Same Edge Weights, 30 points). *Let G be a weighted graph and let T and T' be two different Minimum Spanning Trees of G . Show that the set of weights of edges in T is the same as the set of the weights of the edges in T' . Conclude that if the edges of G all have distinct weights that there is a unique minimum spanning tree.*

Hint: If $e \in T$ has a weight not in T' try adding it to T' . This will allow you to improve T' unless there is a cycle containing only e and edges of lighter weight. In the latter case, find a way to improve T .

Proof. Let G be a weighted graph and T, T' be two minimum spanning trees of G . Assume for contradiction that there exists $e \in T$ such that its weight $w(e)$ is not in the set of weight of edges in T' . Add e to T' and call the new graph S . Then, S contains a cycle C which contains e . Suppose the weight of e is not the largest among all edges in C . Let e' be an edge of larger weight in C , and remove it from S , denote the new graph by $\hat{T}'$. Since e' is in a cycle in S , it is not a bridge, and thus $\hat{T}'$ is connected. By the construction, $\hat{T}'$ has $n - 1$ edges, so it is a spanning tree. Also, $w(\hat{T}') - w(T') = w(e) - w(e') < 0$, which is a contradiction to the fact that T' has the minimal weight among all spanning trees. Therefore, e has the largest weight in C . Since every edge f in T' has $w(f) \neq w(e)$, all other edges in C have weight strictly smaller than $w(e)$. Suppose $e = \{u, v\}$. By deleting e from T we get two disjoint trees T_1 and T_2 , one containing u and the other containing v . Let P be C with e deleted. Then P is a path from u to v . Since P connects T_1 and T_2 , there exists $g \in E(P)$ such that $g \cap V(T_1)$ and $g \cap V(T_2)$ are non-empty. Add g to the union of T_1 and T_2 , and denote the new graph by $\hat{T}$. Then, $\hat{T}$ is connected and by construction it has $n - 1$ edges. Since $V(\hat{T}) = V(T_1) \cup V(T_2) = V(T) = V(G)$, $\hat{T}$ is a spanning tree of G . Then, $w(\hat{T}) - w(T) = w(g) - w(e) < 0$, contradiction. Hence, the set of weights of edges in T is a subset of the set of weights of edges in T' . By the symmetry of this problem, they have to be the same set.

Suppose that the edges of G all have distinct weights. Let T and T' be minimum spanning trees of G . Let $e \in E(T)$. Then the weight $w(e)$ of e is in the set of weights of the edges in T , and therefore in the set of weights of the edges in T' . Thus, $e \in E(T')$ because no other edge has the same weight. Therefore, $E(T) \subseteq E(T')$, and they are then equal since they have the same cardinality. Also, since T and T' are spanning trees, $V(T) = V(T') = V(G)$. Thus, $T = T'$. Hence, G has a unique minimum spanning tree. $\square$

Question 3 (Average Number of Leaves, 20 points). *How many trees on vertices labelled $1, 2, \dots, n$ have the vertex labelled k as a leaf? Use this to compute the average number of leaves of a tree with vertices labelled $1, 2, \dots, n$ and show that this number is approximately n/e for large values of n .*

Proof. By Cayley's theorem, there are $(n - 1)^{n-3}$ trees on $n - 1$ vertices labelled $1, \dots, k - 1, k + 1, \dots, n$. For each of those labelled trees, adding a vertex labelled k and an edge $\{k, u\}$ for any $u \in \{1, \dots, k - 1, k + 1, \dots, n\}$ creates a distinct tree on vertices labelled $1, 2, \dots, n$ where the vertex labelled k is a leaf. This construction covers all such trees since removing the vertex labelled k gives a tree on vertices labelled $1, \dots, k - 1, k + 1, \dots, n$. Therefore, the number of such trees equals to the number of ways to produce distinct trees in this construction, which is $(n - 1)^{n-3}(n - 1) = (n - 1)^{n-2}$.

Let $\mathcal{A}$ be the set of all trees with vertices labelled $1, 2, \dots, n$. Then, by Cayley's theorem, $|\mathcal{A}| = n^{n-2}$. For each tree $T \in \mathcal{A}$, let $\text{leaf}(T)$ be the set of labels of the leaves in T . For each $i \in \{1, 2, \dots, n\}$, let $T(i)$ be the set of all trees on vertices labelled $1, 2, \dots, n$ that have the vertex labelled i as a leaf. Then, by the last part, $|T(i)| = (n - 1)^{n-2}$ for all i . For each pair $T \in \mathcal{A}$ and $i \in \{1, 2, \dots, n\}$, define $\delta(T, i)$ to be 1 if T has the vertex labelled i as a leaf, and 0 otherwise. Finally, let μ be the average number of leaves of a tree with vertices labelled $1, 2, \dots, n$. Then,

$$\begin{aligned}
\mu &= \frac{1}{|\mathcal{A}|} \sum_{T \in \mathcal{A}} |\text{leaf}(T)| \\
&= \frac{1}{|\mathcal{A}|} \sum_{T \in \mathcal{A}} \sum_{i \in \text{leaf}(T)} 1 \\
&= \frac{1}{|\mathcal{A}|} \sum_{T \in \mathcal{A}} \sum_{i=1}^n \delta(T, i) \\
&= \frac{1}{|\mathcal{A}|} \sum_{i=1}^n \sum_{T \in \mathcal{A}} \delta(T, i) \\
&= \frac{1}{|\mathcal{A}|} \sum_{i=1}^n \sum_{T \in T(i)} 1 \\
&= \frac{1}{|\mathcal{A}|} \sum_{i=1}^n |T(i)| \\
&= \frac{1}{|\mathcal{A}|} \sum_{i=1}^n (n-1)^{n-2} \\
&= \frac{n(n-1)^{n-2}}{n^{n-2}} \\
&= n \left(\frac{n-1}{n} \right)^{n-2},
\end{aligned}$$

and

$$\lim_{n \rightarrow \infty} \frac{\mu}{n} = \lim_{n \rightarrow \infty} \left(\frac{n-1}{n} \right)^{n-2} = \lim_{n \rightarrow \infty} \left(\frac{n}{n-1} \right)^2 \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n} \right)^n = e^{-1}.$$

Hence, μ is approximately n/e for large n . □

Question 4 (Last Caley Edges, 20 points). *Recall that in our proof of Caley's Theorem that when converting a tree on vertices labelled $1, 2, \dots, n$ that we stopped removing leaves when there were two vertices left. Prove that these vertices are the one labelled n and the neighbor of this vertex along the unique path from the vertex labelled n and the vertex labelled $n-1$.*

Proof.

Claim. 1. *If T is a tree, then T has at least 2 leaves.*

2. *If T is a tree with exactly 2 leaves u and v , then T is a path from u to v .*

Proof. Let T be a tree. Let n be its number of vertices and m be its number of leaves. Then, by handshake lemma,

$$2(n-1) = 2|E(T)| = \sum_{v \in V(T)} d_T(v) = \sum_{v \text{ is a leaf}} d_T(v) + \sum_{v \text{ is not a leaf}} d_T(v) \geq m + 2(n-m).$$

Thus, $m \geq 2$.

Suppose that T has exactly 2 leaves u and v . By the equation above, $m = 2$ implies that $d_T(w) = 2$ for any w that is not a leaf. Let P be a path from u to v in T , and assume for contradiction that there is a vertex w that is not on this path. Then, there is a path Q from w to v . Let x be the closet vertex to u on among the vertex on P that is also a vertex of Q . Then, x is in an edge $e \in E(Q) \setminus E(P)$. Since u and v are of degree 1, they are not x . Thus, x has two edges in $E(P) \setminus E(Q)$. Thus, x has degree at least 3, contradiction. Hence, P contains all vertices in T , so it has $n-1$ edges. Therefore, $T = P$ is a path from u to v . □

Since all trees have at least 2 leaves, the vertex labelled n will never be deleted because either it is not a leaf or there is another leaf with smaller label. Consider the vertex labelled $n - 1$. Either it is removed in the algorithm or it is not. If it is never removed, then the claim follows. Suppose it is removed. Then, at the stage where the vertex labelled $n - 1$ is about to be removed, all vertices with label smaller than $n - 1$ are not leaves. Thus, by claim 1, the vertices labelled n and $n - 1$ are the only leaves in the remaining tree. By claim 2, the remaining tree at that stage is a path from the vertex labelled $n - 1$ to that labelled n , which is the unique one in the original tree. Suppose the path is $(v_0, v_1, \dots, v_m)$ where v_0 is the vertex labelled $n - 1$ and v_m is the one labelled n . Then, following the algorithm, we will remove, in order, $v_0, v_1, \dots, v_{m-2}$. Thus, the vertices remained are v_{m-1} and v_m , where v_{m-1} is the neighbor of the vertex labelled n along the unique path from the vertex labelled n and the vertex labelled $n - 1$. $\square$

Question 5 (Extra credit, 1 point). *Approximately how much time did you spend on this homework?*