

Math 154 Homework Solution

Fall 2021

Solution to Homework 2 John Li

This homework is due on gradescope by Friday October 8th at 11:59pm pacific time. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in L^AT_EX is recommended though not required.

Please cite any other students with whom you collaborated on any problems.

Question 1 (Non-Multiple of 3 Cycles and Circuits, 25 points). *Prove that if a graph G has a circuit whose number of edges is not a multiple of 3 then by removing edges from this walk, one can find a cycle whose number of edges is not a multiple of 3.*

Proof. Let C denote the circuit whose number of edges is not a multiple of 3. From Theorem 1.2 proven in class, a circuit contains a cycle, with possible repeated vertices. If C is a cycle (no repeated vertices), then we are done. If C has repeated vertices, for each vertex we can split it up (See Figure 1) and make it into two smaller circuits, and in each smaller circuit this vertex is not repeated. By continuing splitting we can make C into a collection of circuits where each circuit has no repeated vertices, hence C is made into a collection of cycles. Note that this operation doesn't change the number of edges, so the number of edges of C is the sum of the edges of the cycles. So there is at least one cycle whose number of edges is not a multiple of 3. By keeping this cycle and removing the other cycles, we are done.

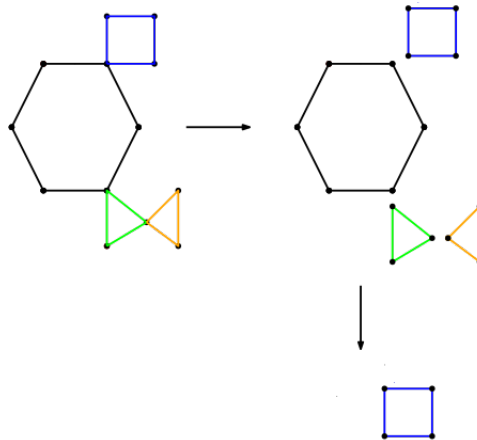


Figure 1: Splitting the repeated vertices and removing the cycles whose number of edges is a multiple of 3.

□

Question 2 (Distances and BFS, 25 points). In a graph G define the distance between two vertices u and v to be the smallest number of edges in any path from u to v (or ∞ if no such path exists). Let G be a connected graph with a vertex u and let T be the breadth first search tree rooted at u . Prove that for every other vertex v in G that the distance from u to v equals the length of the unique path from u to v in T . (In other words the path from u to v in T is a shortest path in G .)

Proof. Since G is connected and T is the spanning tree, v is in T , and hence there is a unique path p that connects u and v . Let's say it has length $l(p)$. By the definition of distance (the shortest length of a uv walk), $d(u, v) \leq l(p)$. If $d(u, v) < l(p)$, there is a shorter uv path outside T , since p is the unique uv path in T . Then this violates the property of Breadth First Search tree, which says that it finds the shortest path from u to v (we prove this in the lemma below). Hence $d(u, v) = l(p)$. $\square$

Lemma Let G be a connected graph and T be the Breadth First Search spanning tree rooted at u , then T finds shortest path from u to other vertices. In other words, if there is a uv path of length k , then the BFS path has length at most k .

Proof. We prove this by induction. When $k = 1$, v is a neighbor of u , which is connected to u in the first step of BFS, hence the length of the BFS path is 1.

Assume the lemma is true for $k - 1$. When the length of the uv path is k , let w be the vertex next to v on this path (see Figure 2), so the length of the uw path is $k - 1$. By inductive hypothesis, the BFS path from u to w has length at most $k - 1$. By the $(k - 1)$ th step of BFS, w will be connected to the tree, and v is a neighbor, hence by k th step v will also be connected to the tree, making the BFS path from u to v has length at most k . $\square$

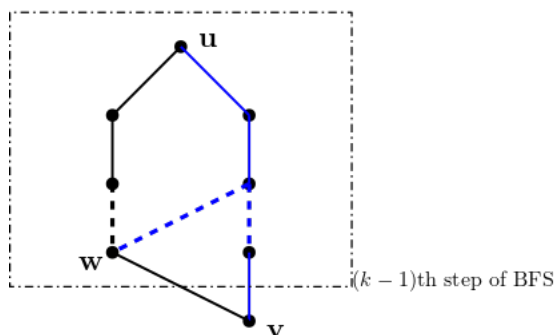


Figure 2: The blue paths are BFS paths. In k th step v must be connected to the tree hence the length of uv path is at most k .

Question 3 (Bridges and Trees, 25 points). Let G be a connected graph. We call an edge e of G a bridge if removing e causes G to become disconnected.

Prove that an edge e of G is a bridge if and only if e is part of every spanning tree of G .

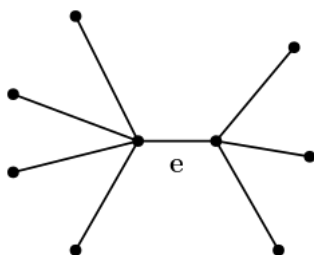


Figure 3: An example of a bridge e .

Proof. " $\implies$ "

If e is a bridge of G , and assume that there is a spanning tree T that doesn't include e . Let G' be the graph G with edge e removed. Then G' is not connected, and so is T since T is the subgraph of G' . This is a contradiction, since a tree is by definition connected.

" $\impliedby$ "

If e is in every spanning tree of G , and assume that e is not a bridge, which means that G' is still connected. As we have shown in class, the connected graph G' will have a spanning tree T , and by definition T is also a spanning tree of G . But T doesn't include e , which is a contradiction. $\square$

Question 4 (Number of Bipartite Colorings, 25 points). *Let G be a finite, bipartite graph with C connected components. How many ways can the vertices of G be colored black and white so that each edge of G connects a black vertex to a white vertex?*

Let's first assume that G is connected, i.e., it has one connected component. Pick an arbitrary vertex u as the starting point. Note that if the color of u is fixed, then the entire coloring of G is fixed, since for any vertex v in G there is a uv path with length l , and v has the same color if l is even and opposite color if l is odd. Hence we have two ways of coloring (u in black or u in white). See Figure 4.

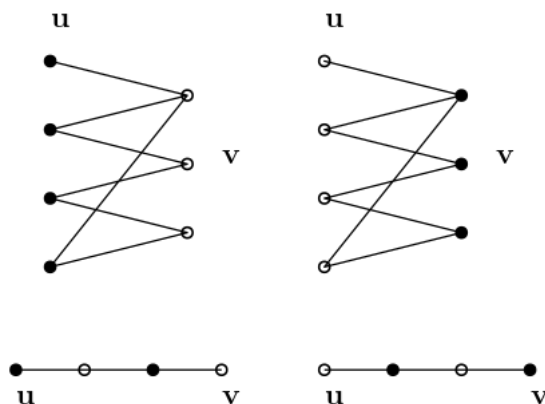


Figure 4: Two ways of coloring a connected bipartite graph.

Now if G has C connected components, note that the connected components are not connected to each other by any edges, any combination of the colorings of the connected components will satisfy the condition. Since each connected component has two ways of coloring, by the multiplication principle, in total G has 2^C ways of coloring.

Question 5 (Extra credit, 1 point). *Approximately how much time did you spend on this homework?*