

Math 154 Homework Solution

Fall 2021

Solution to Homework 1 Haixiao Wang

This homework is due on gradescope by Friday October 1st at 11:59pm pacific time. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in L^AT_EX is recommended though not required.

Please cite any other students with whom you collaborated on any problems.

Question 1 (2-Regular Graphs, 25 points). *Show that any finite, 2-regular graph G is a disjoint union of cycles. In particular, show that G has a number of induced subgraphs that are cycles and so that:*

1. *Each vertex is in exactly one of these induced subgraphs.*
2. *No edges connect these subgraphs to each other.*

Proof. Let $G = (V, E)$ be a finite 2-regular graph with n vertices. For any vertex $v_1 \in V$, there are 2 edges incident to v_1 . Let e_1 be one of them. Following e_1 , we would move to another vertex $v_2 \in V$, which has 2 edges as well since G is 2-regular. Starting at vertex v_2 , one edge e_1 connects it back to v_1 . The other edge e_2 will connect v_2 to some other vertex v_3 . By repeating the process above, we would obtain a path $\{v_1, \dots, v_{n_1}\}$, which must eventually reach a vertex we have seen before. This must be v_1 (since any other v_k is adjacent only to v_{k-1} and v_{k+1}), which gives one cycle C_{n_1} , as an induced subgraph of G . For any vertex $v_k \in C_{n_1}$, if it is connected to an outside vertex $v \in G \setminus C_{n_1}$, the degree of v_k would be 3, since v_k has already been adjacent to v_{k-1} and v_{k+1} , which violates the 2-regularity assumption. Hence C_{n_1} is a connected component of G . Repeating this procedure for each connected component of G , we find that G is a disjoint union of cycles. $\square$

Question 2 (Properties Inherited by Subgraphs, 25 points). *For which of the following graph properties P does the following hold: If G is a graph satisfying P and H is an induced subgraph of G , then H must also satisfy P . For each property, either give a proof or a counter-example.*

- (a) G is a complete graph [5 points]
- (b) G is a bipartite graph [5 points]
- (c) G is a cycle [5 points]
- (d) G is a simple graph [5 points]
- (e) G is a path [5 points]

Proof. Remember that a subgraph H is an induced subgraph if it contains all the edges of G connecting two vertices in $V(H)$.

- (a) If $G = (V(G), E(G))$ is complete, then for any $v \in V(G)$, v is connected to all $u \in V(G) \setminus \{v\}$. As a result, for any $v \in V(H) \subset V(G)$, v is connected to all $u \in V(H) \setminus \{v\}$. Hence $H = (V(H), E(H))$ is complete.

- (b) If $G = (V(G), E(G))$ is bipartite, then the vertex set $V(G)$ can be partitioned into two disjoint subsets $U(G)$ and $W(G)$, i.e., $V(G) = U(G) \cup W(G)$ and $U(G) \cap W(G) = \emptyset$, such that for any $(u, w) = e \in E(G)$, we have $u \in U(G)$ and $w \in W(G)$. Let $H = (V(H), E(H))$ be an induced subgraph of G . Let $U(H) = U(G) \cap V(H)$ and $W(H) = W(G) \cap V(H)$. For any edge (u, w) of H must also be an edge of G and therefore, one of the vertices (say u) is in $U(G)$ and the other (w) in $W(G)$. However, since both are in $V(H)$, this means that $u \in U(H)$ and $w \in W(H)$. Thus, H is bipartite.
- (c) The induced subgraph H may not be cycle. The counter-example can be seen in Figure 1.

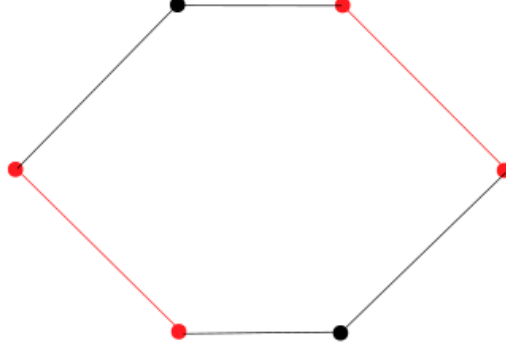


Figure 1: The induced subgraph H of C_6 is marked red, which is not a cycle.

- (d) A simple graph G has neither multiedges, nor self loops. The induced subgraph H doesn't contains any multiedges or self loops. Hence H is simple.
- (e) The counter-example can be seen in Figure 2, where H has 2 disjoint components.



Figure 2: The induced subgraph H of this path is marked red, which is not a path.

□

Question 3 (Hypergraph Handshake Lemma, 25 points). *Suppose that you have a hypergraph where each edge is incident on exactly k vertices. Formulate and prove a version of the Handshake Lemma for this type of graph.*

Proof. Remember that in graph $G = (V, E)$ the degree of vertex $v \in V$, denoted by $d(v)$, is the number of vertices adjacent to v , i.e., the number of edges containing v . A hypergraph $H = (V, E)$ is called k -uniform if each edge $e \in E$ is incident on exactly k vertices. The degree of v , denoted by $d(v)$, can be similarly defined as the number of edges containing v . The analogous Handshake Lemma is

$$\sum_{v \in V} d(v) = k|E|. \quad (1)$$

To prove this, we are going to count the number of vertex-edge incidence pairs in two different ways. On left hand side, each vertex v is incident on $d(v)$ hyper-edges, thus the total number of pairs in this hypergraph is $\sum_{v \in V} d(v)$. On right hand side, each hyper-edge $e = (v_1, v_2, \dots, v_k)$ contains k incident vertices $v_1, v_2, \dots, v_k$, thus the total number of pairs is $k|E|$.

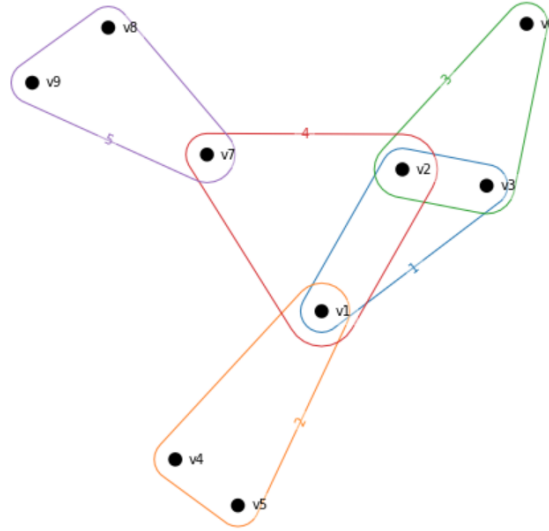


Figure 3: An example of 3-uniform hypergraph H with 9 vertices and 5 hyperedges.

□

Question 4 (3-Regular Graphs, 25 points). *Show that for every even integer $n \geq 4$ that there is a 3-regular graph with exactly n vertices. What happens if n is odd?*

Proof. 1. Let $n = 2k$ for some $k \geq 2$. Consider the cycle C_{2k} on vertices $\{1, 2, \dots, 2k-1, 2k\}$. The desired graph is obtained by connecting vertex i and $i+k$ for all $1 \leq i \leq k$. The degree of vertex i ($1 \leq i \leq k$) is 3 since i is connected to $i-1$, $i+1$ and $i+k$. The degree of vertex i ($k+1 \leq i \leq 2k$) is also 3 since i is connected to $i-1$, $i+1$ and $i-k$. Here vertex 1 is connected to $2k$.

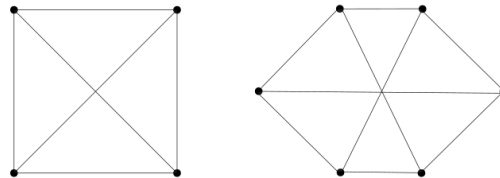


Figure 4: Examples for $n = 4$ and $n = 6$.

2. There is no such graph. The Handshake Lemma, $\sum_{v \in V} d(v) = 2|E|$, indicates that the sum of degrees should be even, however $3n$ is odd when n is odd.

□

Question 5 (Extra credit, 1 point). *Approximately how much time did you spend on this homework?*

Solution to Homework 2