

Question 1 (Edge Counting, 25 points). *If a graph has five vertices of degree 7 and eleven vertices of degree 3, and no other vertices, how many edges in total does the graph have?*

By the Handshake Lemma we have that

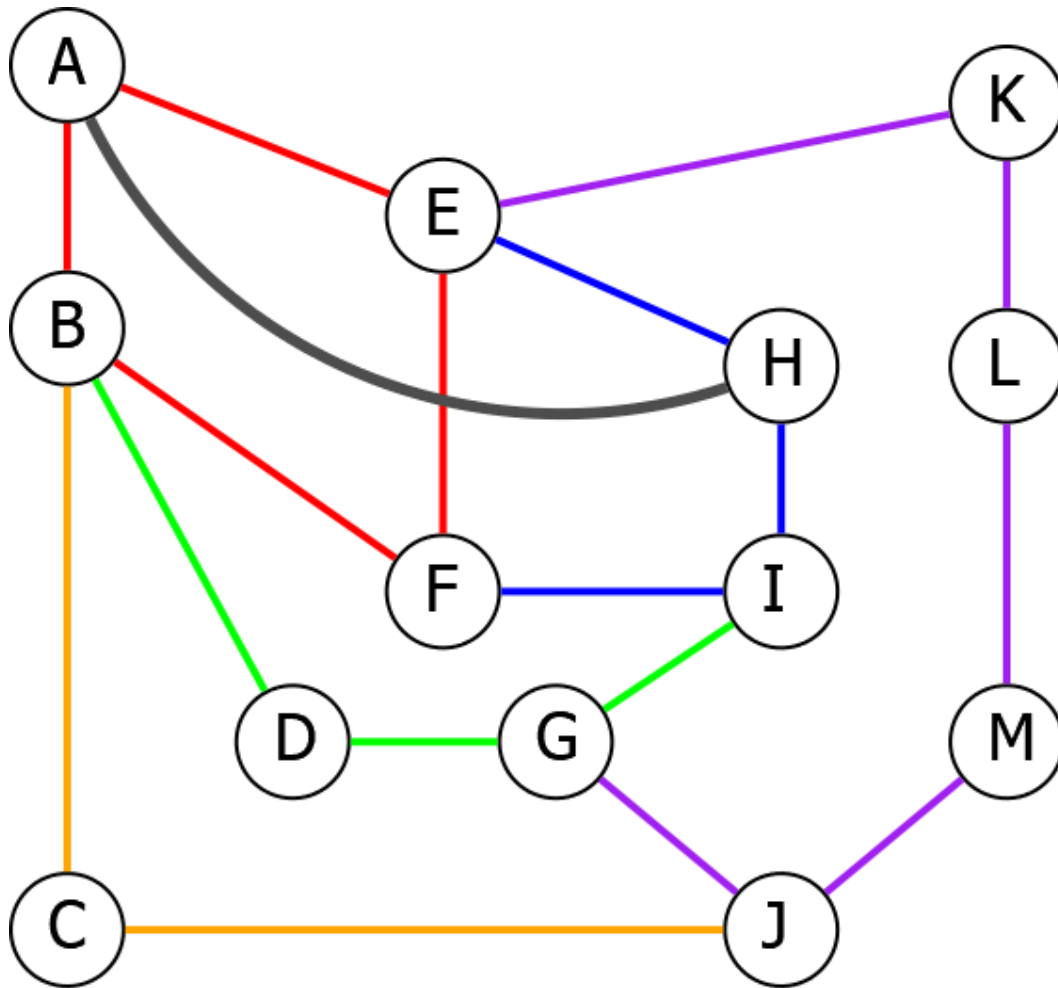
$$2|E| = \sum_v \deg(v) = 5 \cdot 7 + 11 \cdot 3 = 68.$$

Thus, the number of edges is 34.

Question 2 (Triangle-Free Graphs, 25 points). *What is the greatest number of edges that a K_3 -free graph with 21 vertices can have?*

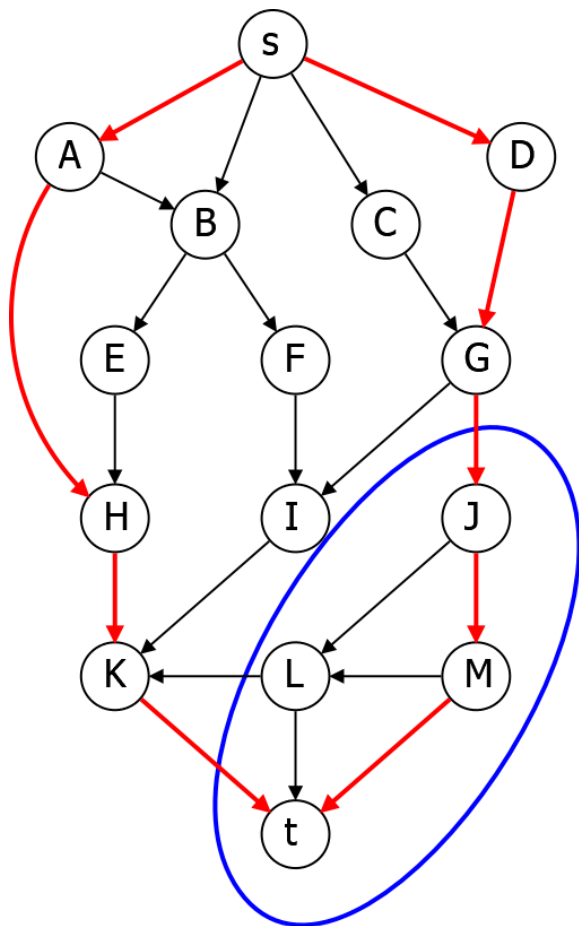
By Turan's Theorem, the greatest number of edges is attained by a $K_{10,11}$, which has $10 \cdot 11 = 110$ edges.

Question 3 (Ear Decomposition, 25 points). *Show how to write the graph below as a cycle plus ears.*



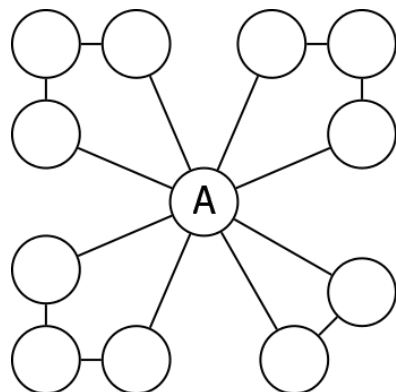
There are many possible solutions, but one is to start with the red cycle and then add the blue, green, purple, orange, and grey ears in that order.

Question 4 (Minimum Cut Finding, 25 points). *Find a minimum cut in the graph below:*



The cut separating t, L, M, J from the other vertices has size 2. This is minimal since the red edges give a flow of the same size.

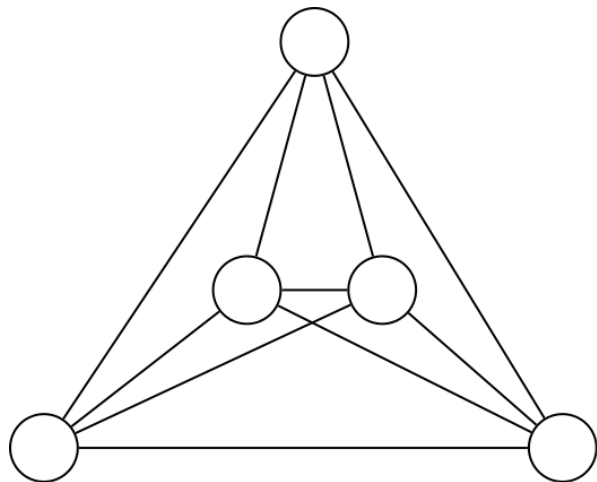
Question 5 (Eulerian Perfect Matchings, 25 points). *Give an example of an Eulerian graph G with an even number of vertices that does not have a perfect matching. Justify your answer.*



The example above works. Every vertex has even degree and it is connected, so it is Eulerian. However, the removal of A leaves three connected components with an odd number of vertices, so it does not have a perfect matching. Alternatively, $K_{2,4}$ has all even degrees, but because the parts have different sizes, it does not have a perfect matching.

Question 6 (Coloring Almost Planar Graphs, 25 points). Call a simple graph G almost planar if it can be drawn in the plane with at most one pair of crossing edges.

(a) Give an example of an almost planar graph G with $\chi(G) \geq 5$.



The above figure shows that K_5 is almost planar. However, it has chromatic number 5.

(b) Show that any almost planar graph G has $\chi(G) \leq 5$.

Suppose that G is an almost planar graph with an embedding where only the edges (a, b) and (c, d) cross. Removing vertex a from the graph, we get a planar embedding of $G - \{a\}$ (since no remaining edges cross). By the Four Color Theorem, we can color $G - \{a\}$ with at most four colors. Coloring a with a fifth color, shows that $\chi(G) \leq 5$.

Question 7 (Leaves Span Trees, 25 points). *Let T be a finite tree with vertices v and w . Show that there is a leaf u so that the path from v to u contains w .*

Let P be the unique path from v to w . Extend this path until it cannot be extended any further. The path will be $v, x_1, x_2, \dots, x_k, w, x_{k+1}, x_{k+2}, \dots, x_\ell, u$. We note that all neighbors of u must have already been used (since we cannot extend the path any further). However, none of the x_i or v or w can be neighbors of u as otherwise $u, x_i, x_{i+1}, \dots, x_\ell, u$ would be a cycle, which one cannot have in a tree. Therefore, u 's only neighbor is x_ℓ , so u must be a leaf.

Question 8 (Ramsey Number Sums, 25 points). *Prove for positive integers n, m and k that $R(n, m+k-1) \geq R(n, m) + R(n, k) - 1$.*

Hint: Start with a coloring of the complete graph on $R(n, m) - 1$ vertices with no red K_n or blue K_m and a coloring of the complete graph on $R(n, k) - 1$ vertices with no red K_n or blue K_k . Combine them to get a coloring of the complete graph on $R(n, m) + R(n, k) - 2$ vertices with no red K_n nor blue K_{m+k-1} .

Let $M = R(n, m)$ and $K = R(n, k)$. By the definition of Ramsey numbers, there is a red-blue coloring of the edges of K_{M-1} so that there is neither a red K_n nor a blue K_m . Similarly, there is a red-blue coloring of the edges of K_{K-1} so that there is neither a red K_n nor a blue K_k . Consider the following red-blue coloring of the edges of K_{M+K-2} . Split the vertices into a K_{M-1} , a K_{K-1} , and the connected edges. Color the edges of the K_{M-1} as specified above, the edges of the K_{K-1} as specified above and all of the connecting edges blue. We claim that this doesn't have a red K_n nor a blue K_{m+k-1} , showing that $R(n, m+k-1) \geq R(n, m) + R(n, k) - 1$.

For red K_n , note that since all connecting vertices are blue, a red K_n would need to be contained either in the K_{M-1} or the K_{K-1} , which by assumption don't have it.

For the blue K_{m+k-1} , it must have either at least m vertices in the K_{M-1} (which implies that the K_{M-1} has a blue K_m), or at least k vertices in the K_{K-1} (which implies that the K_{K-1} has a blue K_k). Since neither of these can happen, there is no blue K_{m+k} . This completes the proof.