

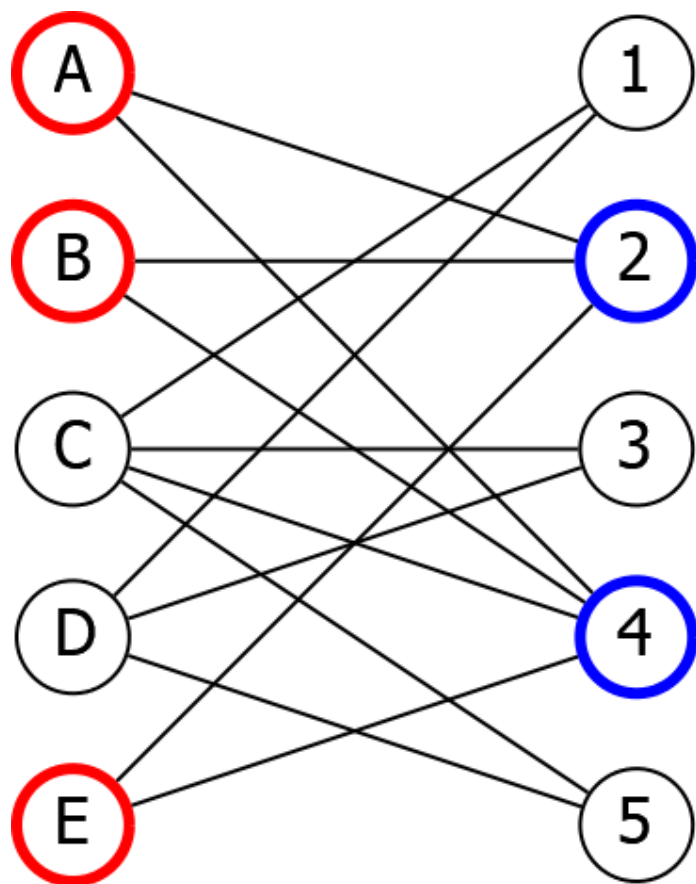
Question 1 (Edges in Planar Graph, 25 points). *Let G be a connected planar graph all of whose faces are pentagons. If G has 62 vertices, how many edges does it have?*

By the face handshake lemma

$$2e = \sum_{\text{faces } f} \text{sides}(f) = 5f.$$

Euler's Formula tells us that $v - e + f = 2$, so we have that $v - (3/5)e = 2$. Noting that $v = 62$, we find that $e = 100$.

Question 2 (Perfect Matching, 25 points). *Find a perfect matching in the graph below or show that none exists.*



Note that in the figure above that A, B , and E are adjacent only to 2 and 4. Thus, no perfect matching can exist.

Question 3 (High Degree Planar Graphs, 25 points). *Let G be a simple, planar graph with $n \geq 5$ vertices, three of which are u, v and w . Show that $\deg(u) + \deg(v) + \deg(w) \leq 2n + 2$.*

We claim that there are at most 2 vertices of G adjacent to all of u, v, w . This is because if there were three such vertices x, y, z , the subgraph induced by u, v, w, x, y, z would contain a $K_{3,3}$ contradicting the assumption that G is planar. We note that the sum of the degrees of u, v , and w is the sum over all vertices x of the number of u, v, w adjacent to x . This is 3 for at most two of the vertices and at most 2 for the rest. Thus, the sum of degrees is at most $2 \cdot 3 + (n - 2) \cdot 2 = 2n + 2$.

Question 4 (Max Edges for Chromatic Number, 25 points). *Let n and k be positive integers. What is the greatest number of edges that a simple graph G with nk vertices and chromatic number at most k can have? Justify your answer.*

The answer is $k(k-1)n^2/2$. This can be achieved by splitting the vertices into k groups of size n and drawing edges between each pair of vertices not in the same group. This graph is k -colorable by coloring each group a different color.

To show that this is optimal, we note that if a k -coloring of G has n_i vertices of the i^{th} color then we have $n_1 + n_2 + \dots + n_k = nk$. Furthermore, G cannot have any edges between vertices of the same color. Thus, the greatest number of edges that G can have is the number of pairs of vertices not in the same group. This is at most $(nk)(nk-1)/2 - \sum_i n_i(n_i-1)/2$. It is not hard to see that this is maximized when $n_i = n$ for all i .