Math 154 Exam 2 Solutions

Spring 2020
Question 1 (Polyhedron Faces, 25 points). Suppose that $P$ is a polyhedron with five faces meeting at every vertex, each face being either a triangle or a square. If $P$ has 72 vertices, how many faces of each type does it have?

By the Handshake Lemma, we have that $2|E| = \sum_v d(v) = 5|V| = 360$. Thus, $|E| = 180$. By Euler’s formula, we have that $v - e + f = 2$ so $f = 2 + e - v = 110$. On the other hand, if we have $T$ triangular faces and $S$ square faces $110 = S + T$. The dual handshake lemma says $360 = 2|E| = \sum \text{Sides}(f) = 3T + 4S$. Solving, we have that $S = 30, T = 80$. 

Question 2 (Coloring, 25 points). *Give a 3-coloring of the graph below:*

Many answers are possible, for example:

![Graph coloring example](image-url)
**Question 3** (Straight Line Embedding, 25 points). *Provide a straight line planar embedding of the graph below:*

The graph below is one possible solution.
**Question 4** (Chromatic Numbers and Subgraphs, 25 points). Let $G$ be a finite graph with a complete subgraph $H$. Suppose that $G - H$ is split into connected components $E$ and $F$. Prove that $\chi(G) = \max(\chi(H \cup E), \chi(H \cup F))$, where $H \cup E$ and $H \cup F$ denote the induced subgraphs on the relevant sets of vertices.

Let $k = \max(\chi(H \cup E), \chi(H \cup F))$. On the one hand, any coloring of $G$ restricts to colorings of $H \cup E$ and $H \cup F$, which means that $\chi(G) \geq k$. On the other hand, by assumption we have colorings of $H \cup E$ and $H \cup F$ with only $k$ colors each. Since $H$ is a complete subgraph, each coloring must assign the vertices in $H$ distinct colors. By renaming the colors in the coloring of $H \cup F$, we can make the colors of the vertices in $H$ agree with those in the coloring of $H \cup E$ and so that the other colors used are the same. Once these colorings use the same list of $k$ colors and agree on how they color $H$, we can combine them to make a $k$-coloring of $G$ simply by coloring each $v \in G$ the color that it would be assigned in either the coloring of $H \cup E$ or $H \cup F$. It is easy to check that this coloring works, and thus that $\chi(G) \leq k$. 