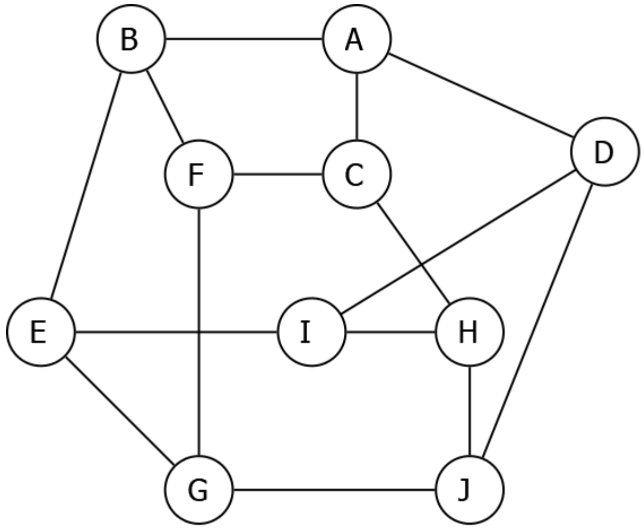
Math 154 Exam 2 Solutions

Spring 2020

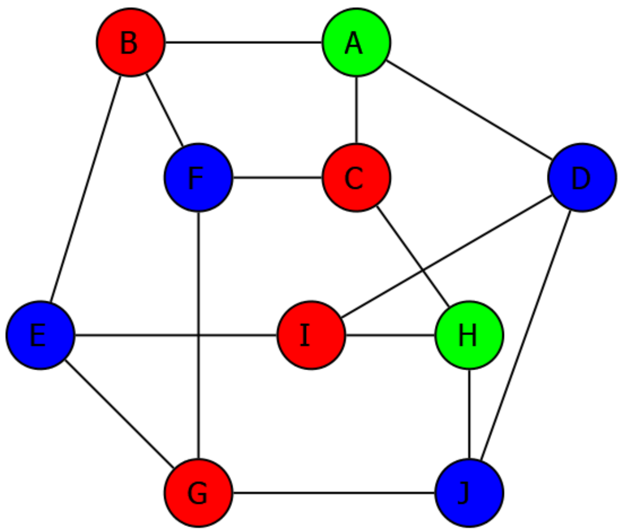
Question 1 (Polyhedron Faces, 25 points). *Suppose that P is a polyhedron with five faces meeting at every vertex, each face being either a triangle or a square. If P has 72 vertices, how many faces of each type does it have?*

By the Handshake Lemma, we have that $2|E| = \sum_v d(v) = 5|V| = 360$. Thus, $|E| = 180$. By Euler's formula, we have that $v - e + f = 2$ so $f = 2 + e - v = 110$. On the other hand, if we have T triangular faces and S square faces $110 = S + T$. The dual handshake lemma says $360 = 2|E| = \sum \text{Sides}(f) = 3T + 4S$. Solving, we have that $S = 30, T = 80$.

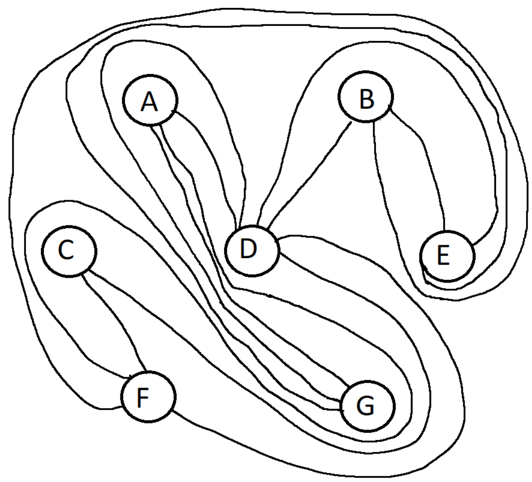
Question 2 (Coloring, 25 points)**.** *Give a 3-coloring of the graph below:*



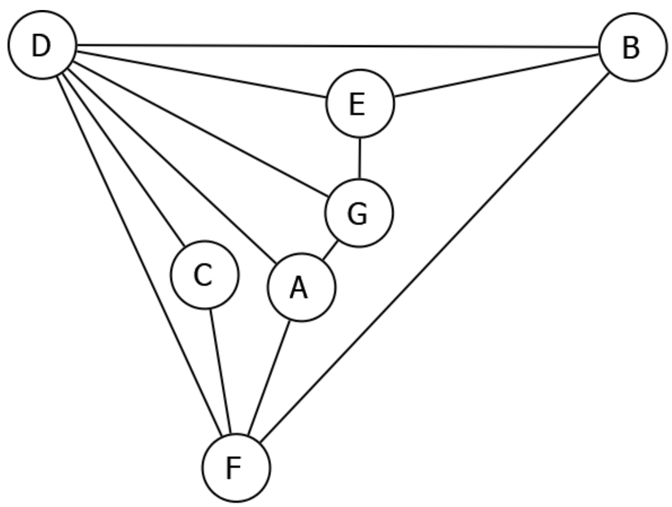
Many answer are possible, for example



Question 3 (Straight Line Embedding, 25 points). *Provide a straight line planar embedding of the graph below:*



The graph below is one possible solution.



Question 4 (Chromatic Numbers and Subgraphs, 25 points)**.** *Let G be a finite graph with a complete subgraph H. Suppose that $G - H$ is split into connected components E and F. Prove that $\chi(G) = \max(\chi(H \cup E), \chi(H \cup F))$, where $H \cup E$ and $H \cup F$ denote the induced subgraphs on the relevant sets of vertices.*

Let $k = \max(\chi(H \cup E), \chi(H \cup F))$. On the one hand, any coloring of G restricts to colorings of $H \cup E$ and $H \cup F$, which means that $\chi(G) \geq k$. On the other hand, by assumption we have colorings of $H \cup E$ and $H \cup F$ with only k colors each. Since H is a complete subgraph, each coloring must assign the vertices in H distinct colors. By renaming the colors in the coloring of $H \cup F$, we can make the colors of the vertices in H agree with those in the coloring of $H \cup E$ and so that the other colors used are the same. Once these colorings use the same list of k colors and agree on how they color H, we can combine them to make a k-coloring of G simply by coloring each $v \in G$ the color that it would be assigned in either the coloring of $H \cup E$ or $H \cup F$. It is easy to check that this coloring works, and thus that $\chi(G) \leq k$.