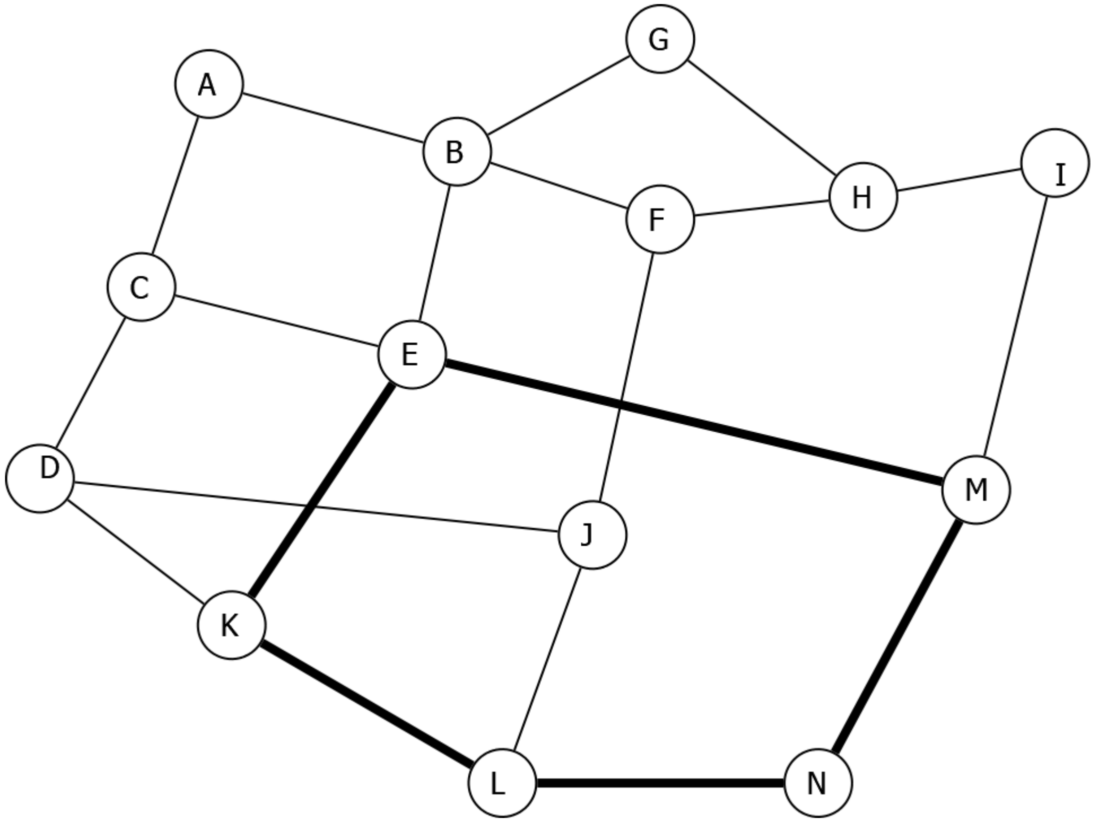
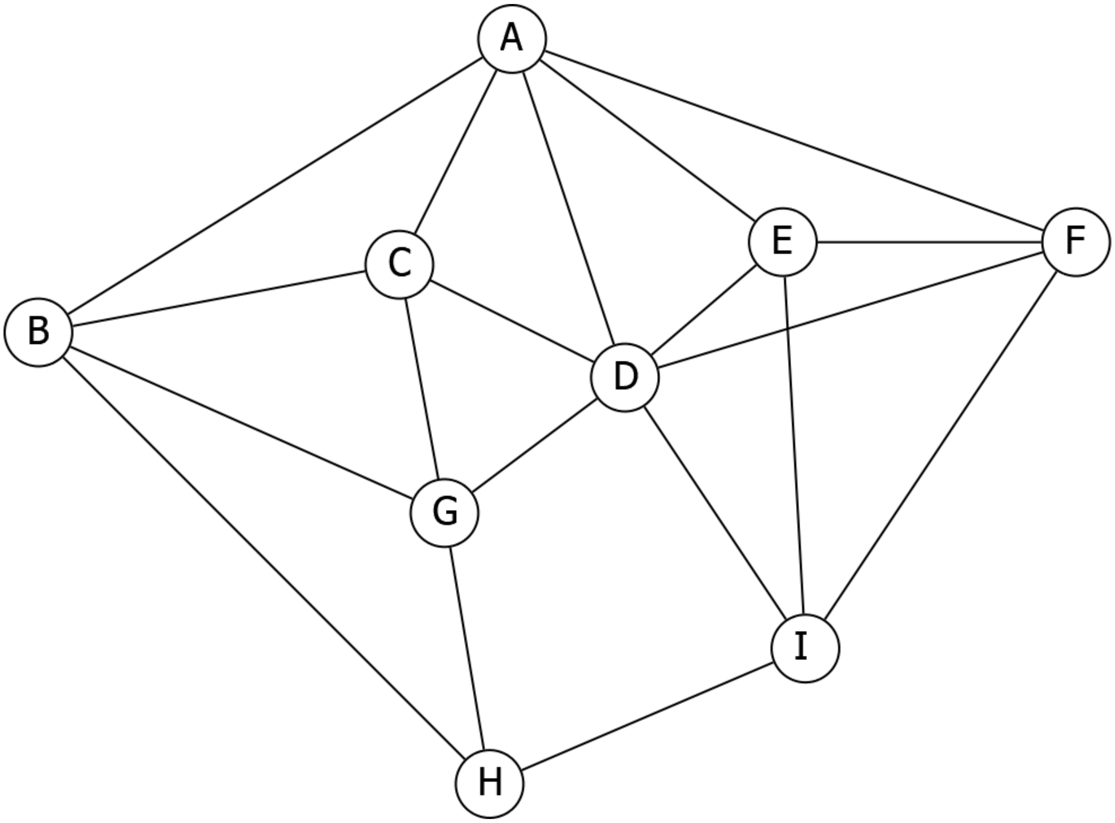
Question 1 (Bipartite Graph, 25 points). *For the graph G below either give a partition of the vertices to make G a bipartite graph or show that no such partition exists.*



Note that the highlighted cycle $EKLMN$ has length 5, which is odd. Therefore the graph cannot be bipartite.

Question 2 ((Semi-)Eulerian Graph, 25 points)**.** *For the graph G below either provide an example of an Eulerian trail in G or show that none exists.*



One example of such a trail is given by $ABCADCGBHGDEAFEIFDIH$. Though many other possibilities exist. (Though all must start/end at A and H.)

Question 3 (Trees and High Degree Vertices, 25 points)**.** *Show that for any tree T on $n \geq 2$ vertices that the number of leaves of T is more than the number of vertices of degree at least* 3*.*

By the Handshake Lemma we have that

$$\sum_{v \in V} d(v) = 2|E| = 2(n-1) = 2n - 2.$$

We note that the sum of the degrees is equal to $1 \cdot n_1 + 2 \cdot 2_n + 3 \cdot n_3 + \ldots$ where n_i is the number of vertices of degree equal to i. This is at least $n_1 + 2n_2 + 3n_{\geq 3}$ where $n_{\geq 3}$ is the number of vertices of degree at least 3. Noting that $n_2 = n - n_1 - n_{\geq 3}$ this gives

$$2n - 2 \geq n_1 + 2(n - n_1 - n_{\geq 3}) + 3n_{\geq 3}.$$

Rearranging this inequality yields

$$n_1 \geq n_3 + 2.$$

Thus, the number of leaves (i.e. vertices of degree-1) is more than the number of vertices of degree at least 3.

Question 4 (Blocks in Semi-Hamiltonian Graphs, 25 points)**.** *For a finite connected graph G show that if G has a Hamiltonian Path that its Block graph must be a path.*

Consider a Hamiltonian Path P in G. For each edge of P write down the unique block of G that contains that edge, and for each cut vertex in P write down that vertex. Then, we have a sequence of blocks and cut vertices. We claim that after removing repeated entries coming from traversing several edges of the same block in a row, this will give us a path in the block graph.

Firstly, we show that this is a walk. From a cut vertex, on other side of it in the path P must be an edge that is part of a block that contains this vertex. Therefore the adjacent entries of our list will correspond to adjacent vertices of the block graph. On the other hand, when we are traversing edges of a given block in G in order to switch to traversing edges of another block, we must move through a vertex at the intersection of these blocks, which must be a cut vertex. Therefore, an element of our list adjacent to a block must be a cut vertex contained in that block (and therefore an adjacent element of the block graph). This proves that our list is a walk. To show that it is a path, we note that since P is a path it can never return to a cut vertex, so all of those entries appear only once. Furthermore, since P is Hamiltonian each time it passes through a cut vertex, it *must* switch from one side of the cut to the other. This is because if it does not switch, it will never be able to get back to the other side and reach the vertices there. Therefore, after this list moves from a block B to a cut vertex v, the path must switch to the opposite side of v and then can never come back to B. Thus, this list is a path in the block graph.

However, it is easy to see that this path must cover the entire block graph. This is because P must reach every cut vertex and every block. However since the block graph is a tree, this path must be the entire block graph. Hence, our block graph is a path.