Announcements

- Homework 2 Due Sunday
- Exam 1 will be on Friday, May 1st if you cannot take it during class time that day (3-4pm pacific time), please email me with the times you can make by Tuesday.
- Reminder: My office hours are still held at the old zoom meeting, not the new one for lecture.

Last Time

Eulerian Graphs

- Eulerian Circuit/trail uses each edge exactly once

Theorem: A connected graph G is Eulerian if and only if every vertex is of even degree.

Question: Equivalent Conditions

For G a finite, connected graph, which of these is equivalent to G being Eulerian?

A) G contains a circuit

B) Every vertex of G has even degree

C) The edges of G can be partitioned into disjoint circuits

D) Every vertex of G is part of a circuit

E) G contains a circuit using all edges

Today

- Semi-Eulerian graphs
- Another algorithm for finding Eulerian circuits

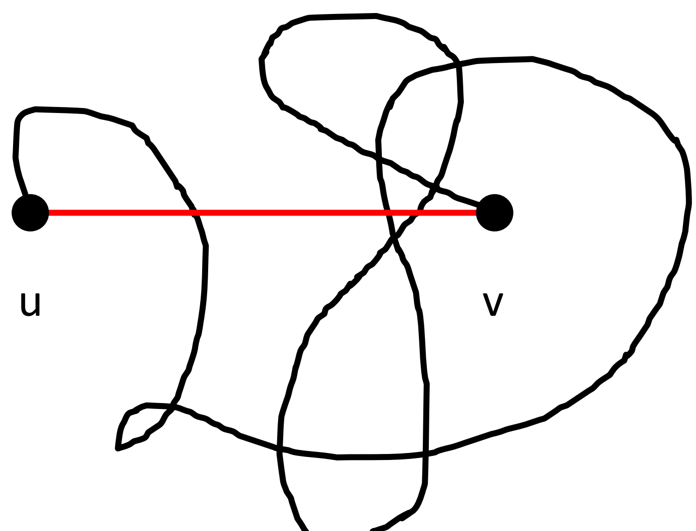
Recall

A graph G is *semi-Eulerian* if it has an Eulerian trail but not a circuit.

What about Semi-Eulerian?

Suppose you have an Eulerian trail starting at u and ending at v.

- If you add an edge from v to u, it becomes a circuit
- If G semi-Eulerian, G+edge is Eulerian.



Necessary Conditions

If G is semi-Eulerian then:

- G must be connected (except for isolated nodes)
- All vertices except for 2, must have even degree (and these must be the endpoints of the path).

Sufficiency

If G has exactly 2 vertices of odd degree:

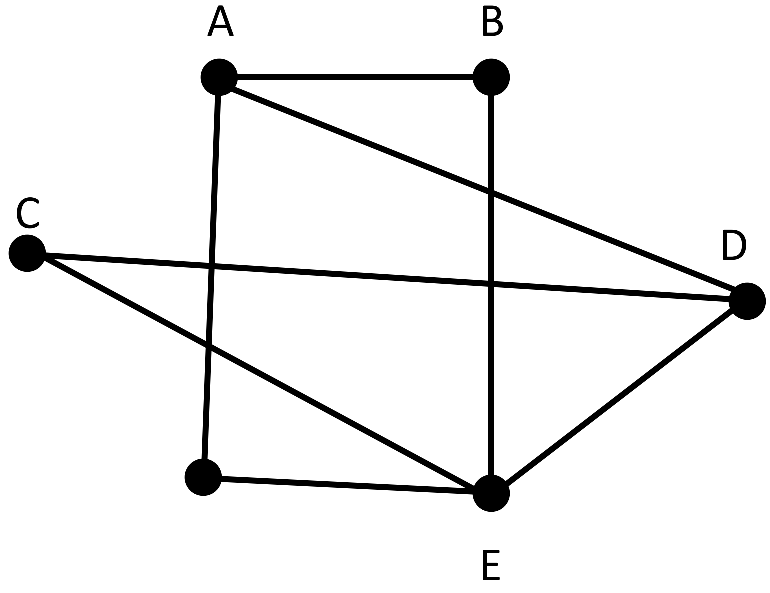
- If u and v odd degree, G+(u,v) is Eulerian.
  - Remove edge (u,v) from circuit and get Eulerian path in G.

Result

Theorem: A finite, connected graph G is semi-Eulerian if and only if it has exactly two vertices of odd degree. Furthermore, these vertices will be the endpoints of any Eulerian trial.

Question: Endpoints

Which vertices are endpoints of an Eulerian trail in the graph below?



New Algorithm

Suppose we've chosen the first few edges of our path, ending at a vertex v. We still need to cover the edges in some subgraph H.

This is possible if and only if

- H is connected (except for isolated vertices).
- H has either all even degrees, or only v and one other vertex are odd degree.

Idea: use this to build a path.

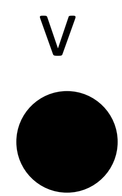
Idea

If H connected and at most one odd degree vertex other than v, pick an edge from v so that after traversing it, this is still true.

Prove by induction on number of edges that if G is connected and either all degrees are even or v and one other are of odd degree that there's an Eulerian trail/circuit starting from v.

Base case: |E| = 0

We're already done!

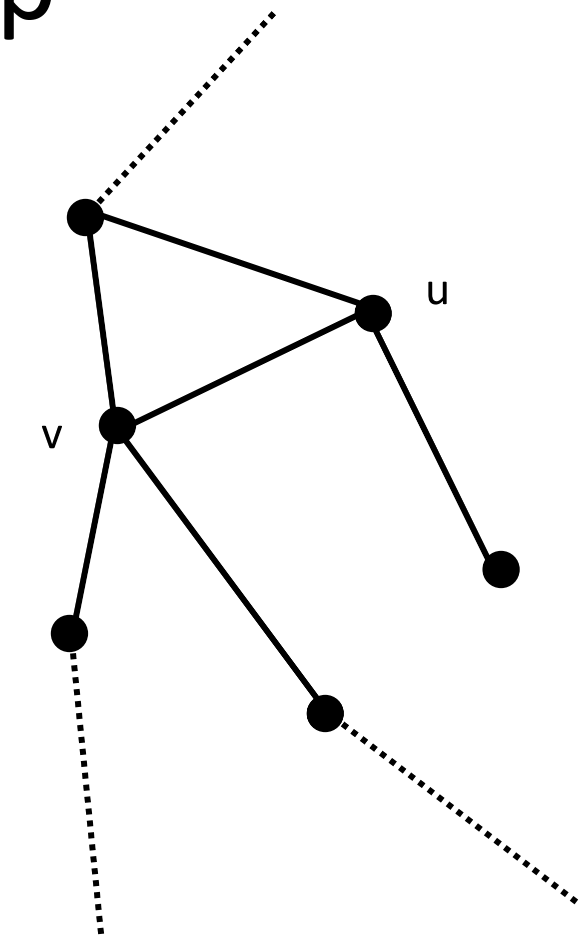


Inductive Step

Suppose it holds for graphs with one fewer edge.

Want: first edge e=(v,u) so that G' = G – e:

- G' is connected
- Either all degrees are even, or only u and one other vertex are odd degree.

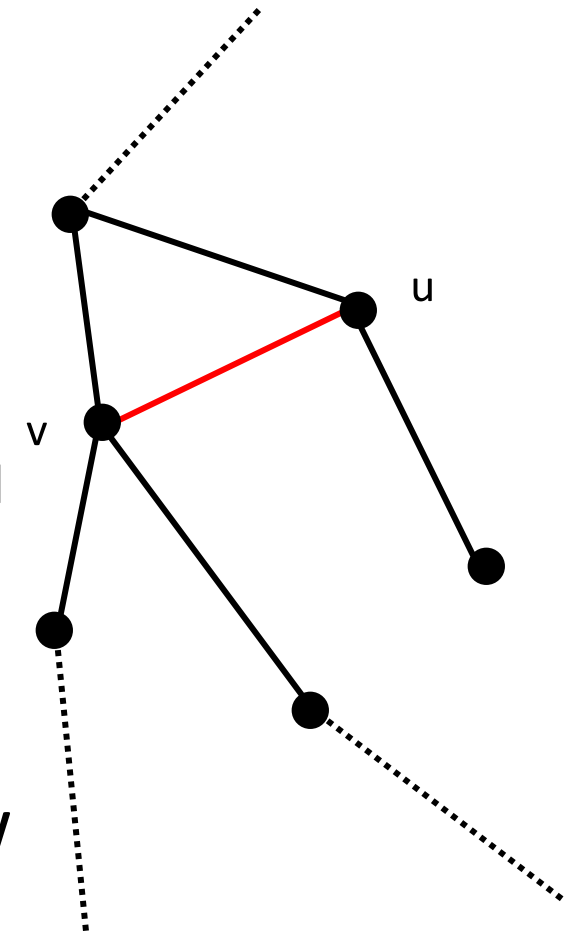


Degree Considerations

Want: either all vertices of H even degree or only u and one other vertex odd degree.

Happens automatically!

- Removing e decreases degrees of u and v by 1, others stay same.
- If all even before, now u, v odd.
- If v and w odd before, now u and w odd.

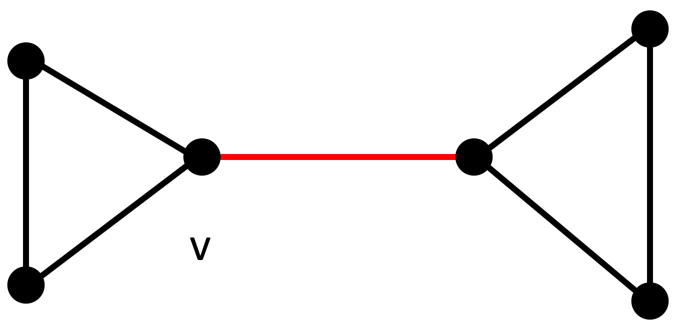


Connectivity

Need to pick an edge so that G’ stays connected.

Not automatic!

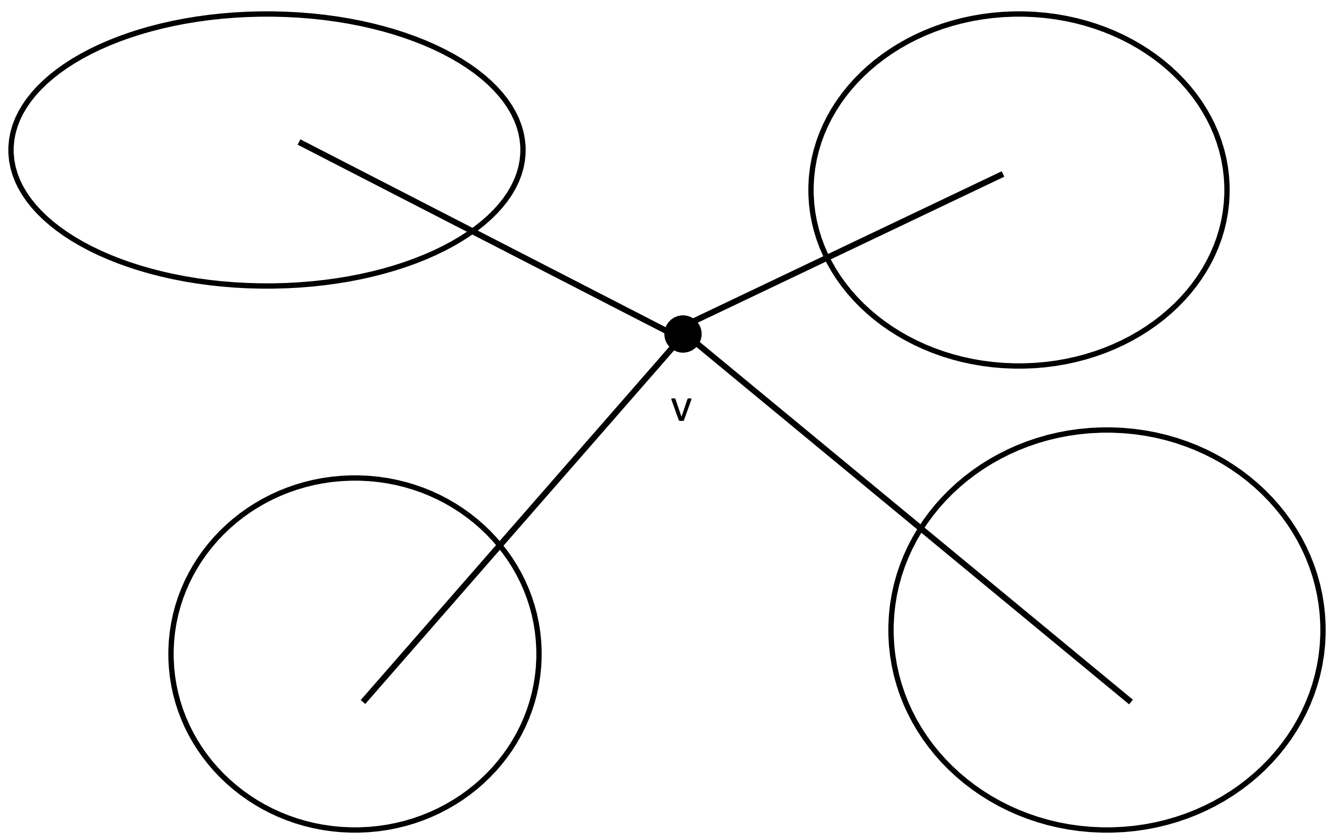
Need to select edge which is not a bridge.



Does such an edge exit?

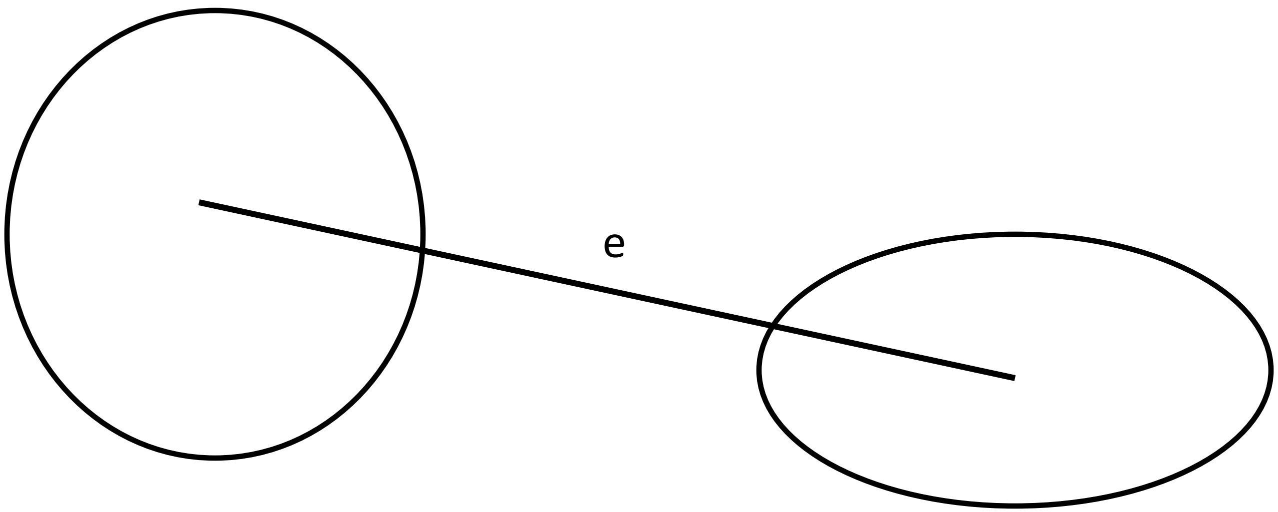
The Bad Case

This is what we want to avoid, is it possible?



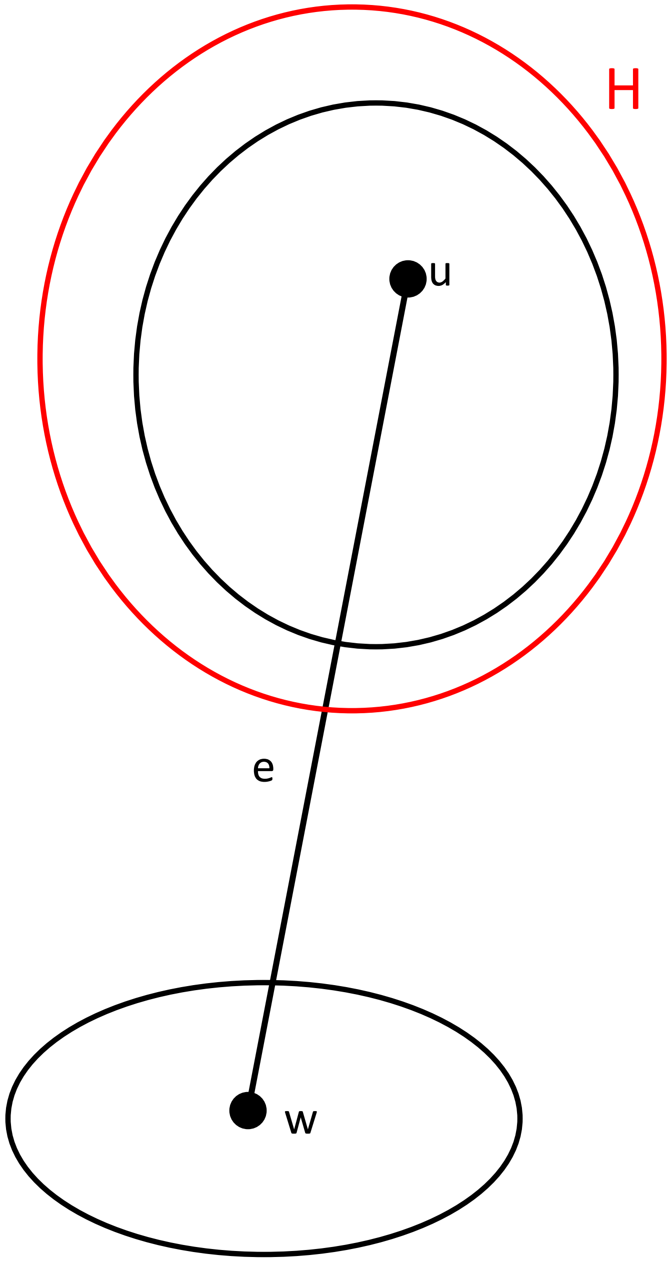
Bridges and Odd Degrees

Lemma: If e is a bridge in a finite graph G, then there is at least one vertex of odd degree on each side of it.



Proof

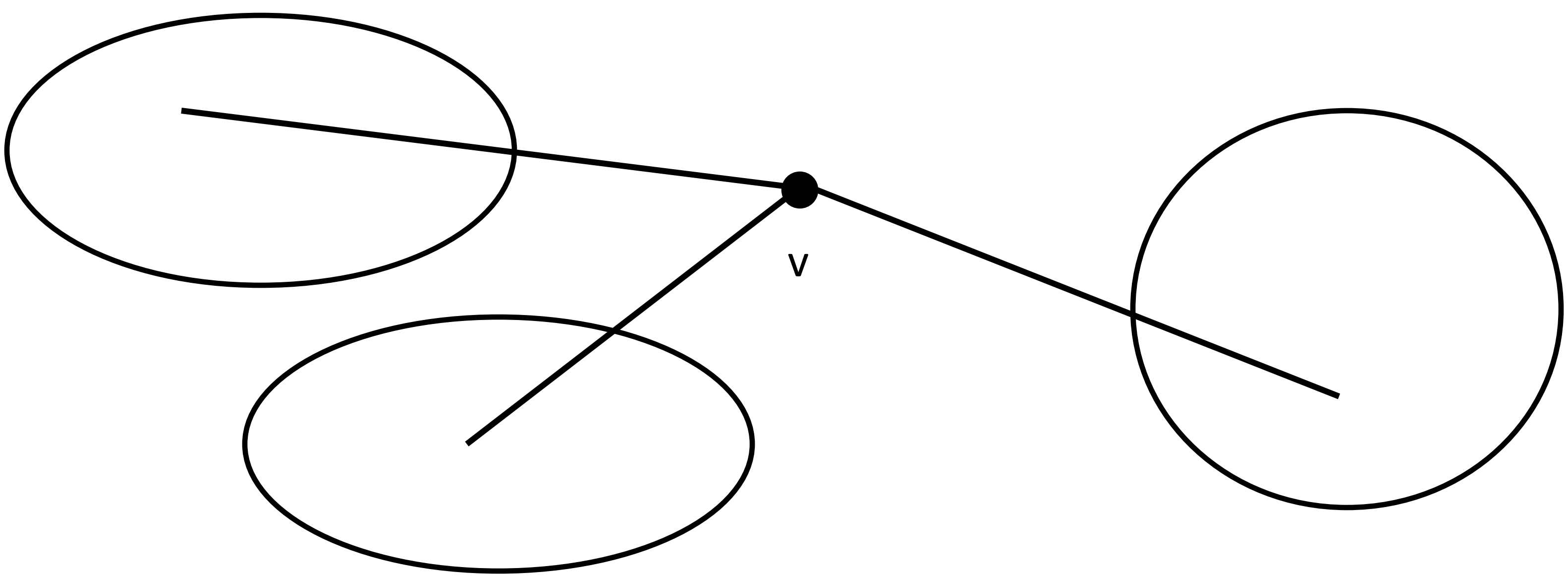
- Let H be the subgraph on one side of the bridge
- Apply The Handshake Lemma to H
  - $\Sigma\, d(v) = 2|E| =$ even
  - Even number of odd deg vertices.
- e increases d(u) by 1
- Counting that, H has odd # of odd deg vertices
- Odd numbers are ≥ 1!



The Bad Case

So this case is impossible.

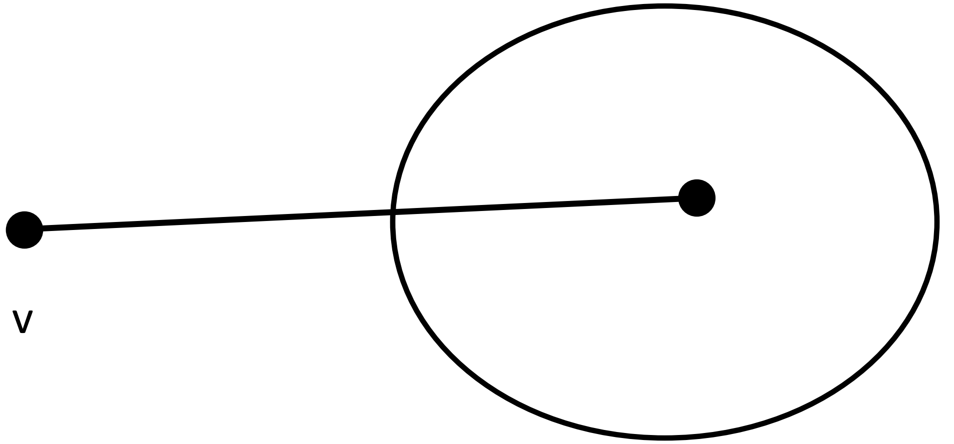
- Each bridge has odd degree vertex on other side
- At most one other odd degree vertex
- Can only happen if d(v) = 1.



Degree 1

What if v has degree 1?

- Following the edge, does disconnect v.
- But it is an isolated vertex, so it's OK.



Review

Either:

- $d(v) = 1$
  - Following edge leaves isolated vertex
- $d(v) > 1$
  - Must be a non-bridge
- Follow appropriate edge, conditions hold for G-e
- By inductive hypothesis can finish our trail

Question: Usable Edges

While constructing an Eulerian trail, you are at a vertex v with d remaining outgoing edges. At least how many of them will allow you to complete the trail?

A) 0

B) 1

C) d-1

D) d