Announcements

- Homework 2 Due Sunday

- Students interested in problem solving practice should attend discussion sections.

Today

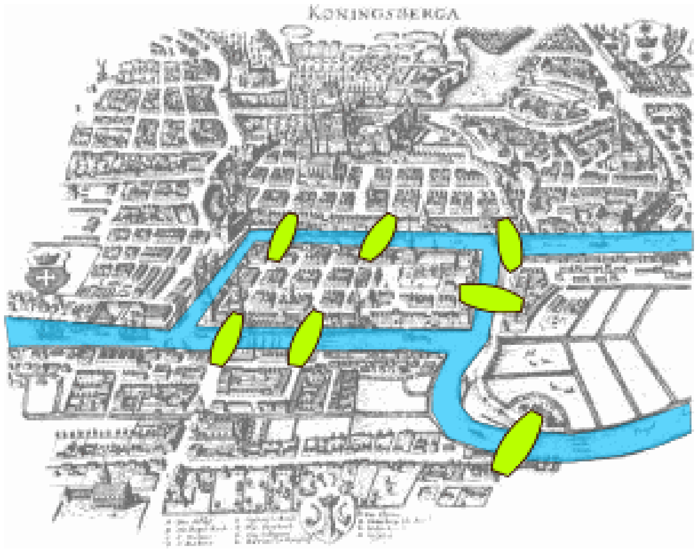
- Start Chapter 1.4
  - Introduction to Eulerian graphs.

Paths and Cycles (Ch 1.4)

- Eulerian Circuits
  - Definition
  - Classification of Eulerian graphs
  - Algorithms
- Hamiltonian cycles
  - Definition
  - Hardness
  - Some conditions

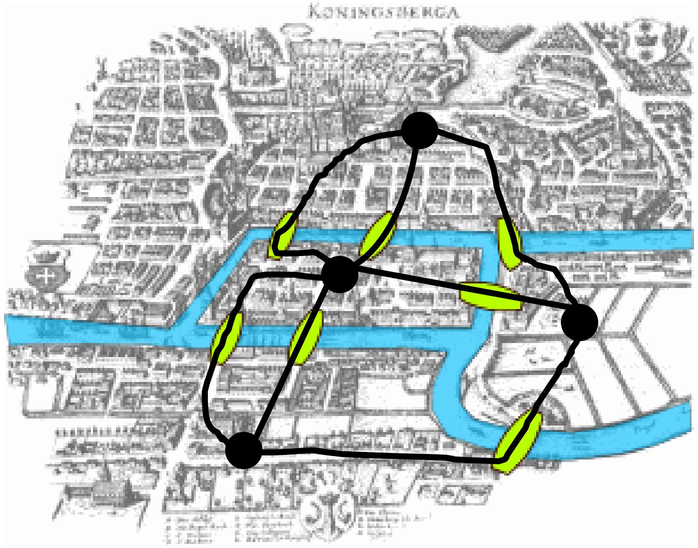
The Bridges of Konisberg

The city of Konisberg had 7 bridges as shown. People liked to go on walks touring the bridges. The mathematician Euler was asked if there was a tour that crossed each bridge exactly once.



Graph Theory

- Turn city into a (multi)graph.
  - Vertices = land areas
  - Edges = bridges
- Want: A walk that uses each edge exactly once.



Stuck!

Definitions

An *Eulerian circuit* is a circuit that uses every edge of a graph exactly once.

An *Eulerian trail* similarly uses each edge exactly once, but does not start and end at the same vertex.

A graph is *Eulerian* if it contains an Eulerian circuit and *semi-Eulerian* if it contains an Eulerian trail.

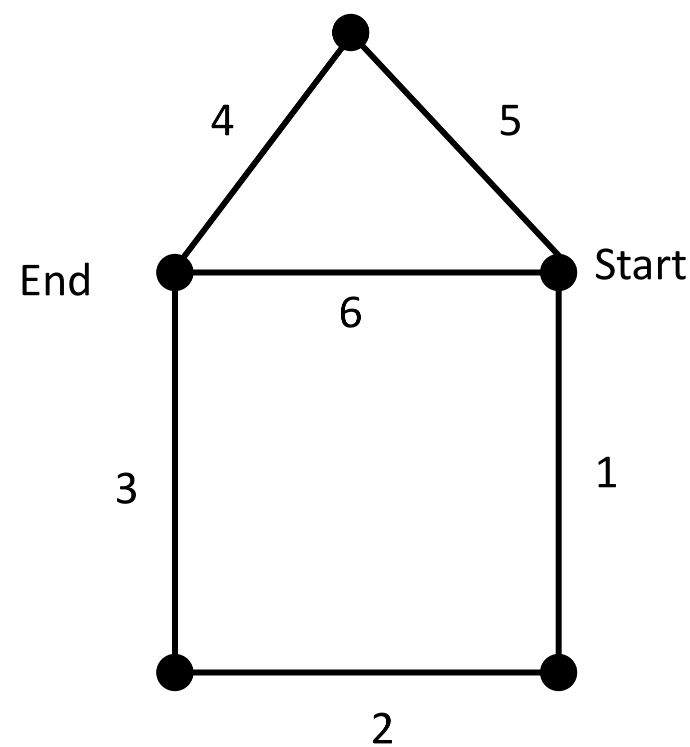
Question

The graph to the right is:

A) Eulerian

B) Semi-Eulerian

C) Neither



Questions we want to answer

- Which graphs are Eulerian / semi-Eulerian?
- How do we construct Eulerian circuits/trials?

Observation I

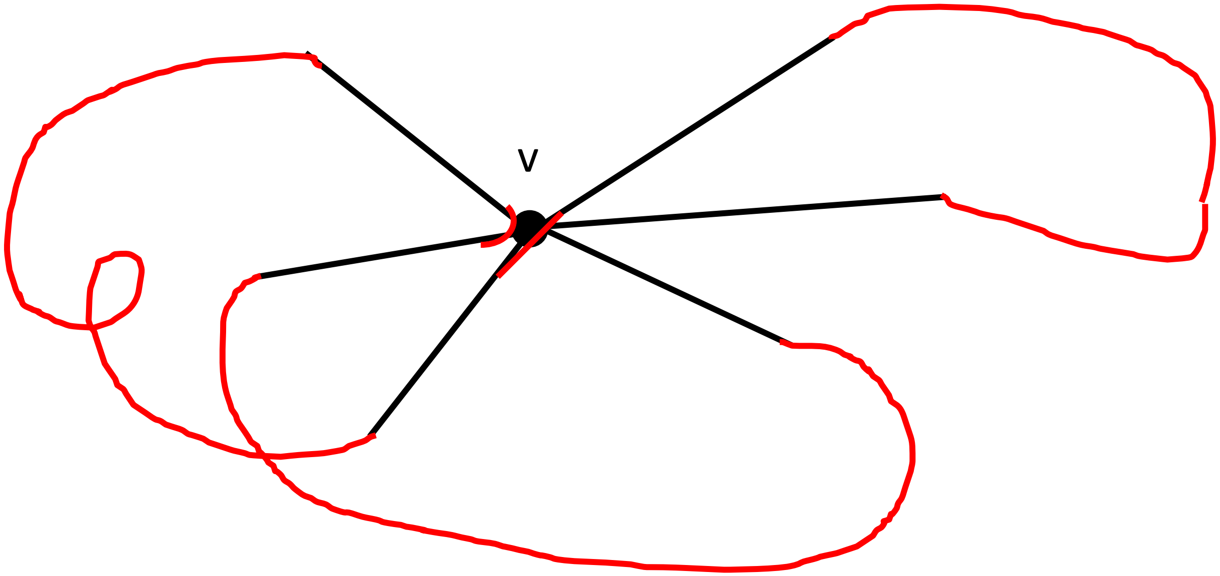
<u>(semi-)Eulerian graphs must be connected!</u>

(except for isolated vertices)

If there is no path from u to v, certainly no Eulerian circuit/trail that connects both of them either.

Observation II

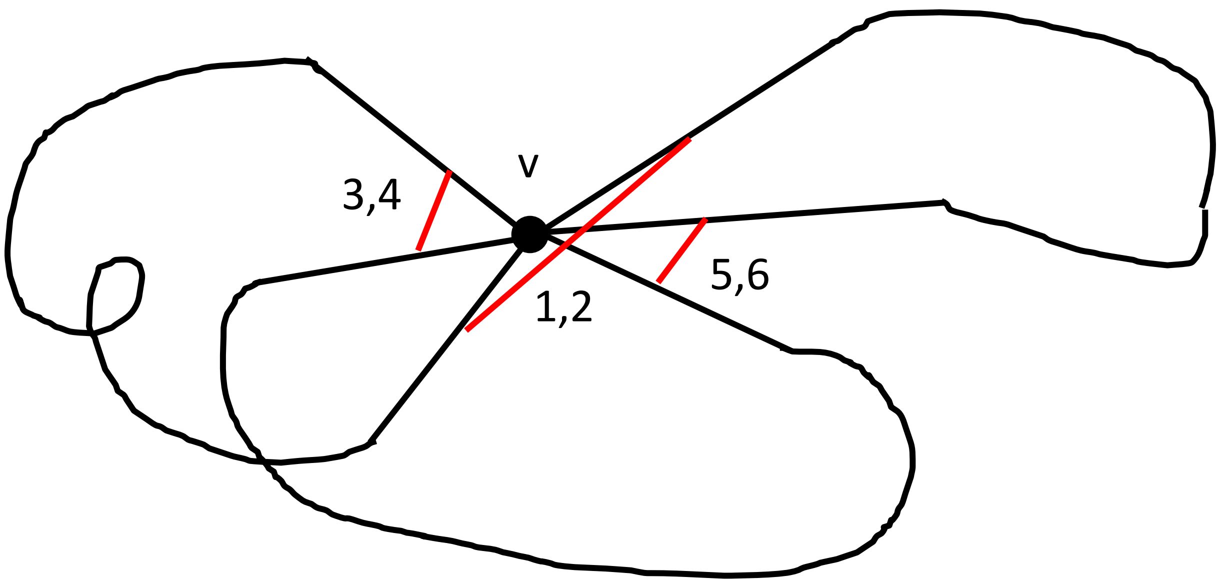
How does Eulerian circuit interact with vertex v?



Each time we take edge *into* v, then take different edge *out of* v.

Observation II

If G is Eulerian, deg(v) must be even!



Conclusion

So if G is Eulerian then it must:

- Be connected (except for isolated vertices)
- Have all vertices of even degree

Is this enough?

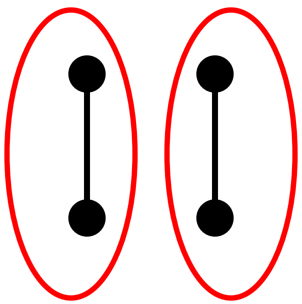
Yes!

Theorem (1.20): A finite, connected graph G is Eulerian if and only if all vertices have even degree.

Question: Eulerian Graphs

Which of these graphs are Eulerian?

A



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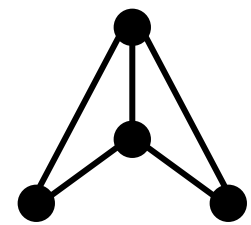

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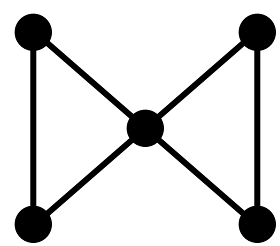

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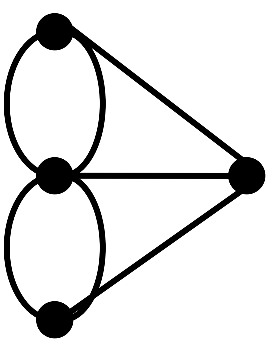

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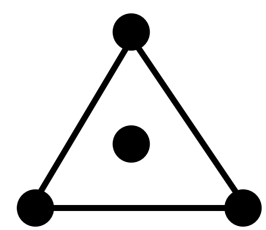


Proof

- "Only if" already done.
  - If G has Eulerian circuit, each time the circuit passes through v it uses two of its edges.
  - Since Eulerian, eventually uses all of v's edges
  - Therefore, deg(v) must be even.
  - This must hold for all vertices v.
- Need to prove that this is sufficient
  - Show how to construct an Eulerian circuit.

Constructing a Circuit I

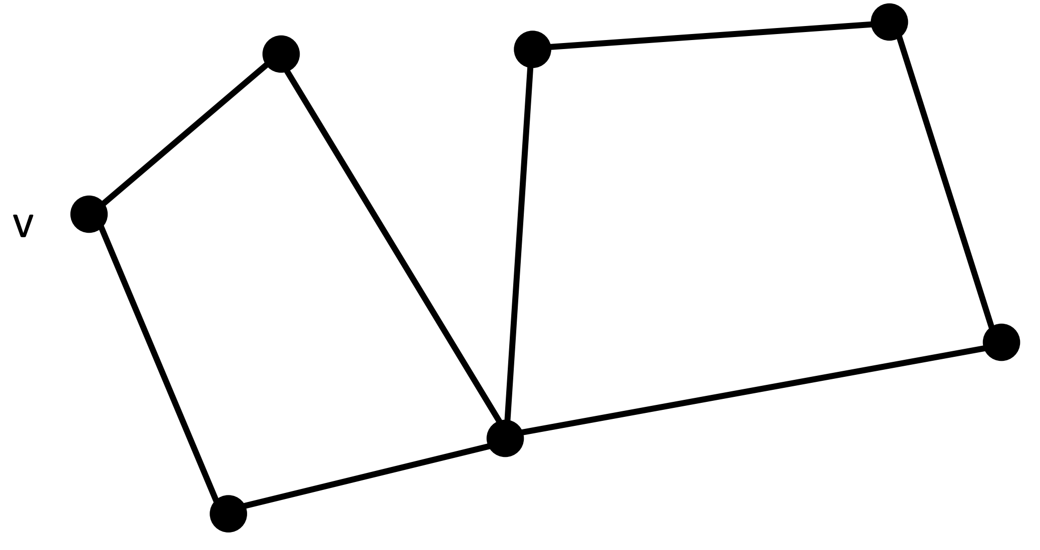
As a first step, we will show you can construct *a* circuit.

Basic facts:

- Every degree is even
- Every non-isolated vertex has degree at least 2.

Constructing a Circuit II

- Start at any non-isolated vertex v.
- Construct a trail by adding new edges until you get stuck.



Constructing a Circuit III

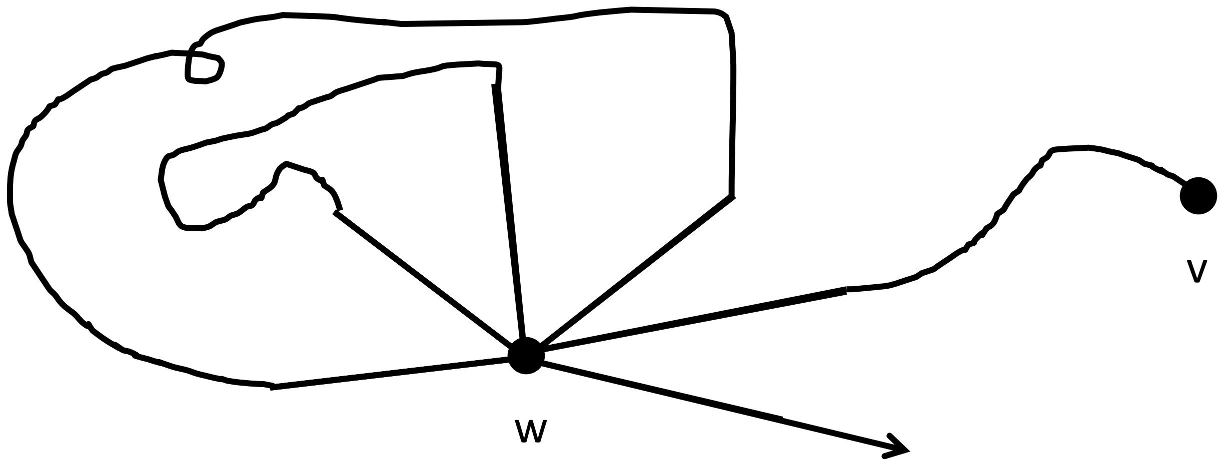
Claim: Can *only* get stuck at v.

Once you get stuck, you will have a circuit!

Proof:

- Suppose you got stuck at some other w.
- Each time you pass through w, use up 2 edges.
- Takes another edge to reach w. If at w used an odd number of edges.
- At least one left!

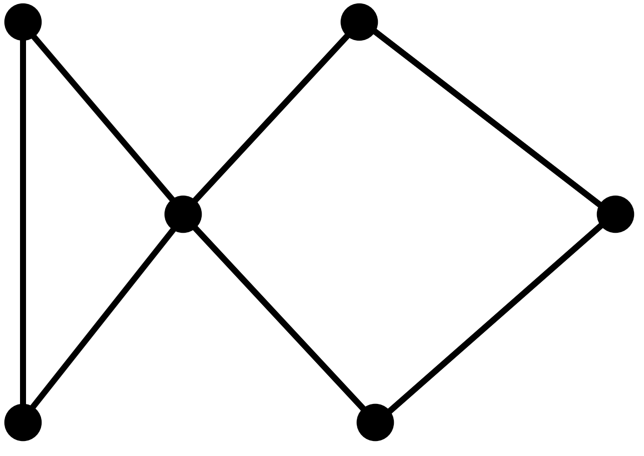
Constructing a Circuit IV



Are we done?

We have *a* circuit. Does it necessarily cover all the edges?

No.



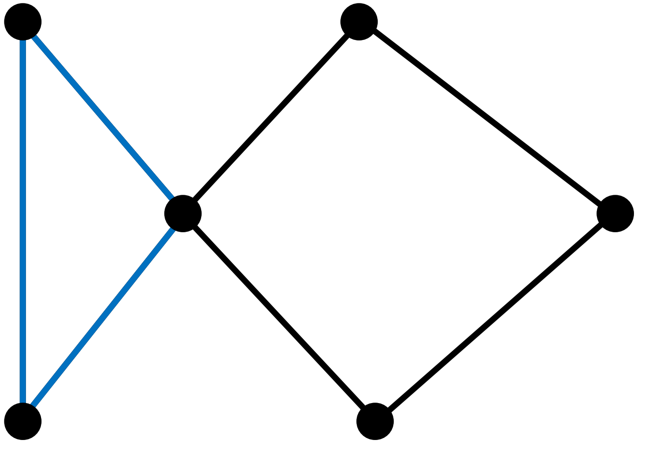
How do we fix this?

Two ideas:

- Can find more circuits
- Glue circuits together

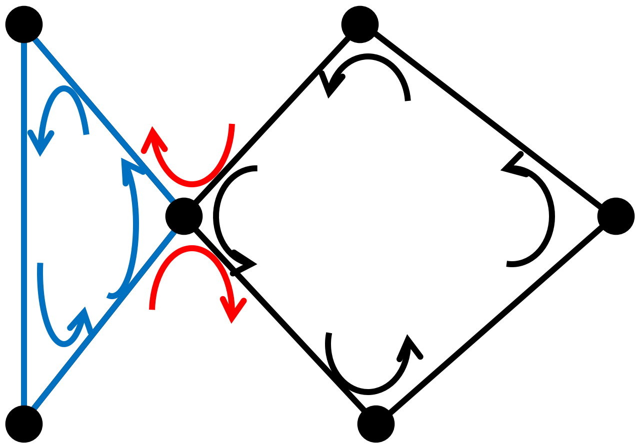
More Circuits

- Existing circuit uses an even number of edges at each vertex.
- Removing those edges, have an even number left at each vertex.
- Can create new circuit in remainder.



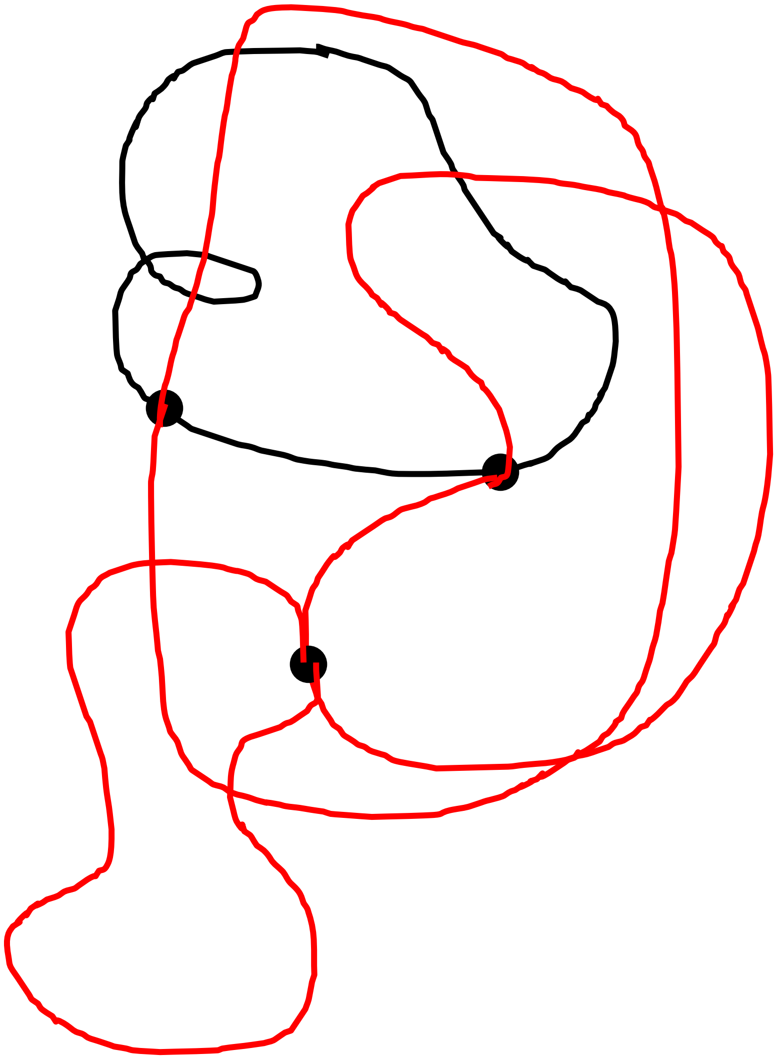
Combining Circuits

- Have two circuits that share a vertex.
- Turn them into one big circuit.



Final Algorithm

- Find a circuit.
- If all of G -> done.
- Otherwise, find v on circuit with unused edge.
- Find additional circuit through v.
- Merge with existing circuit.
- Repeat



Analysis

- By connectivity, if circuit isn’t all of G, contains some vertex with an extra edge. (Otherwise your circuit would be a full connected component)
- Each step increases the number of edges in your circuit. Eventually, you must get all of G.