Announcement

Homework 1 Due Sunday

Reminder: OH's

Daniel: Thursday 12-1, Friday 12-2, or by appointment, https://ucsd.zoom.us/my/dankane

Ji: Monday, Wednesday 1:00-2:30 https://ucsd.zoom.us/my/jzeng

Jiaxi: Tuesday, Thursday 4:00-5:30pm https://ucsd.zoom.us/j/6511860878

Last Time

Trees:

Graph that is

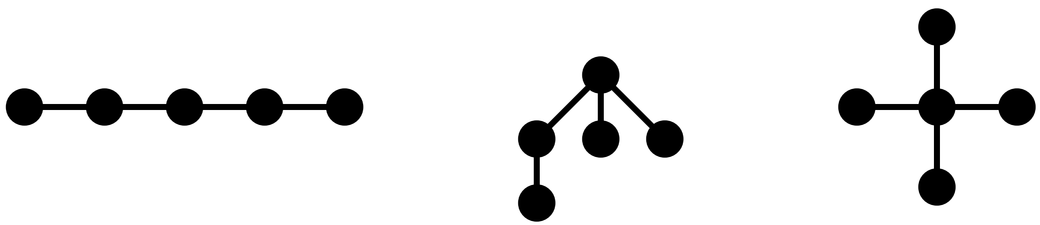
- Connected
- Has no cycles

Today

- Number of Edges in a Tree
- Spanning Trees

5-Vertex Trees

Here are all the trees on 5-vertices:



Notice that they all have exactly 4 edges. This is not a coincidence.

Edge Count

Theorem (1.10): Any tree with n vertices has exactly n-1 edges.

Idea: Look at connected components.

Lemma: Any graph $G = (V,E)$ with no cycles has $|V| - |E|$ connected components.

Proof Idea

Idea: Each edge you add connects two different components, thus reducing the number of connected components by 1.

We proceed by induction on the number of edges in G.

Base Case: |E|=0

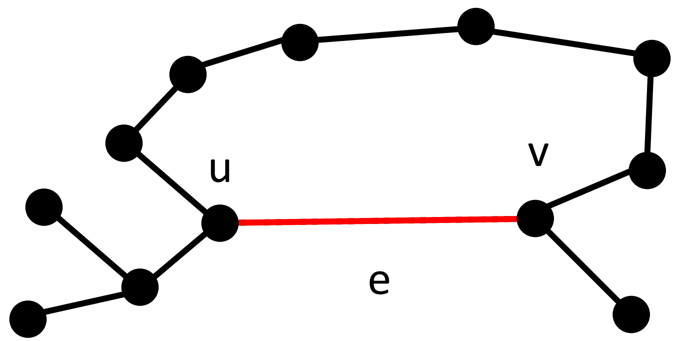
- G is the empty graph.
- Each vertex is a connected component.
- #CCs = #Vertices = |V| = |V|-|E|.

Inductive Step

- Assume that for any graph with fewer edges that #CCs = |V|-|E|.
- Pick any edge e = (u,v) in G.
- Consider G' = G-e.
- #CC(G') = |V| - (|E|-1) = |V|-|E|+1.

Claim

Claim: u and v are in different connected components of G'.

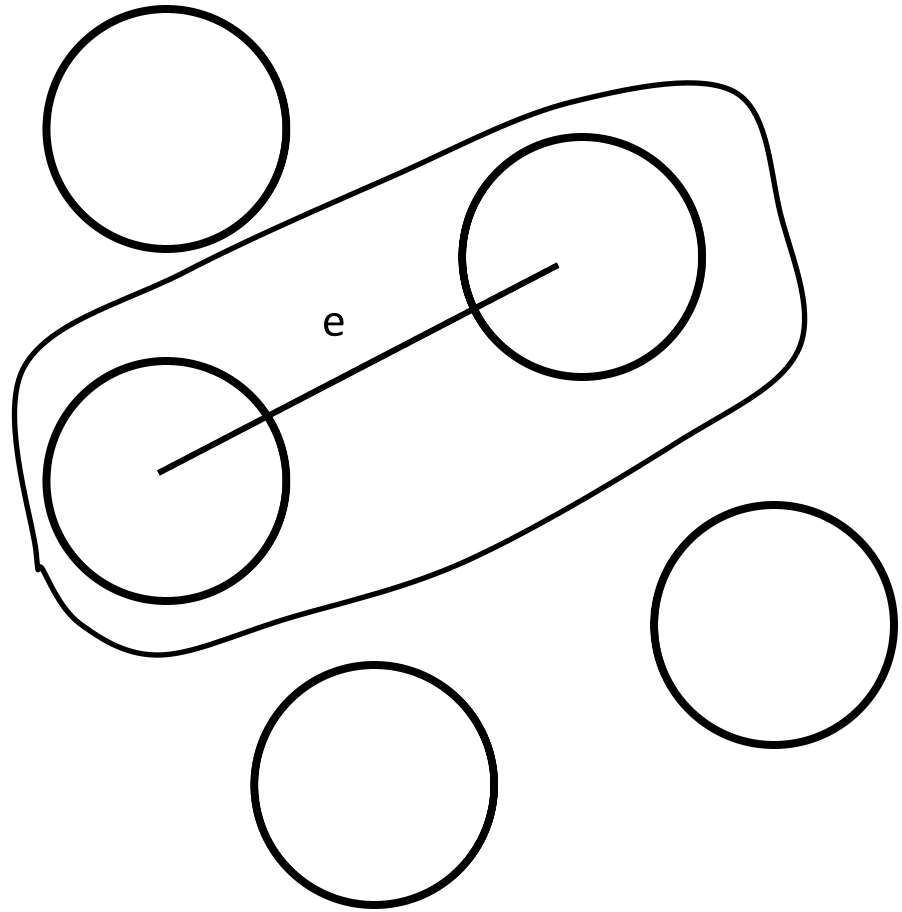


Assume for sake of contradiction, they are in same component.

- Must be a u-v path in G'
- Adding e gives a cycle in G
- Contradiction!

Putting it Together

- G’ has |V|-|E|+1 components
- e connects two *different* components of G’
- Components of G are *same* as those of G’ except two are merged
- G has |V|-|E| CCs



Question: Graphs with Cycles

Suppose graph G has at least one cycle. What can we say about the number of connected components?

A) #CCs < |V| - |E|
B) #CCs = |V| - |E|
C) #CCs > |V| - |E|
D) Impossible to tell

If a new edge forms a cycle, it increases |E| *without* decreasing the number of CCs.

Leaves

A *leaf* in a tree is a vertex of degree 1.

Lemma (Thrm 1.14): Any tree on $n > 1$ vertices has at least two leaves.

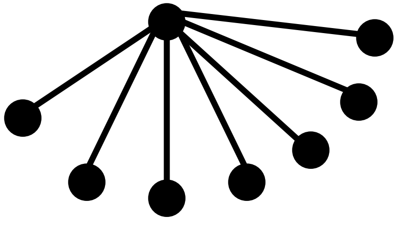
Proof: Handshake Lemma:

- $\Sigma\, d(v) = 2|E| = 2n-2$
- $\Sigma(2-d(v)) = 2n-(2n-2) = 2$
- Leaves contribute 1, non-leaves at most 0
- At least 2 leaves.

Question: Maximum Leaves

What is the *greatest* number of leaves a tree on n>2 vertices can have?

A) 0
B) 2
C) n-1
D) n
E) None of the above



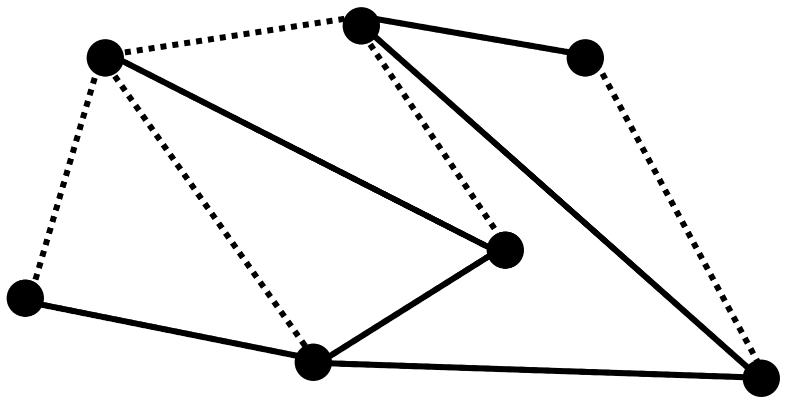
Spanning Trees

Recall our highway repair problem

Want a subgraph T of G so that:

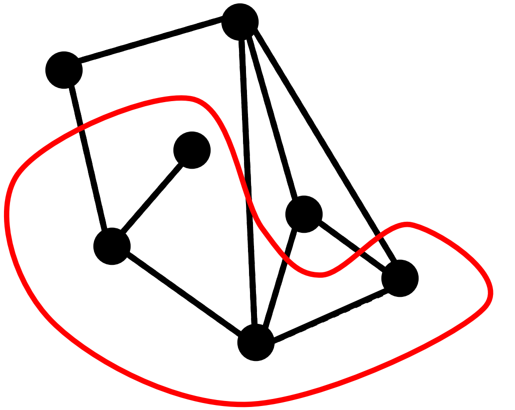
- V(T) = V(G)
- T is a tree

Such a T is called a *spanning tree* of G.



Construction

Does a connected graph G always have a spanning tree? Yes!



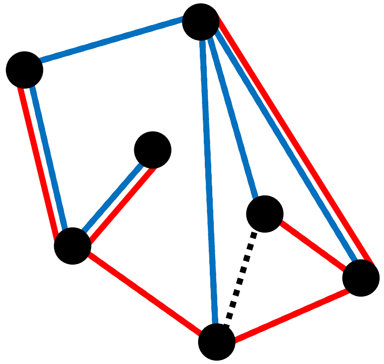
<u>Algorithm:</u>

1) Start with no edges in T
2) While T not connected, add an edge connecting different components
3) Once T is connected, you have a tree

Since G is connected, it must have edges between different components of T

Types of Spanning Trees

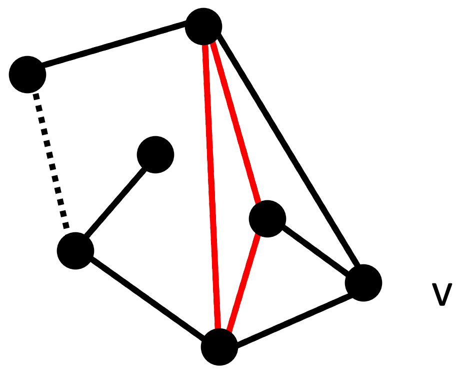
This algorithm leaves you with a lot of freedom. You can pick *any* edge that doesn't create a cycle at any stage. This leaves many possible spanning trees.



We'll discuss a few standard ways to produce spanning trees.

Breadth First Search Tree

- Start at a base vertex v
- Connect v to all its neighbors
- Connect them to all their neighbors (without creating cycles)
- Repeat until you've reached all vertices



Breadth First Search Properties

- Finds *shortest* paths from v to other vertices.
  - d^{th} round of edges finds all vertices reachable from v with paths of length d.
- No edges in G provide shortcuts from a vertex to its descendants further down.