

Last Time

- Terminology
 - Neighborhood
 - Degrees
 - Regular graphs
 - Handshake Lemma
- Types of graphs
 - Paths
 - Cycles
 - Complete Graphs
 - Bipartite Graphs
 - Subgraphs

Today

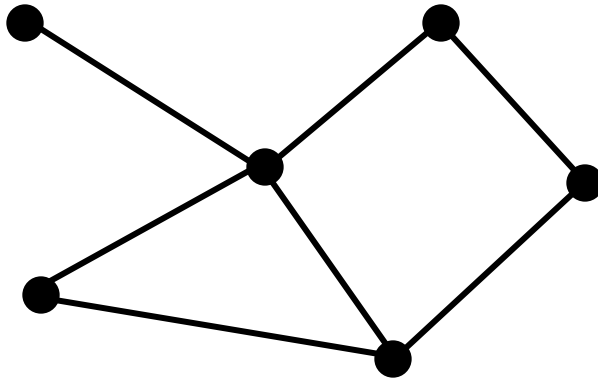
- Walks, paths, and connectivity
- Path reduction
- Classification of Bipartite Graphs

Walks, Paths, and Connectivity

Many applications of graph theory talk about some kind of network where travel along edges makes sense. This is formalized using the concept of a *walk*.

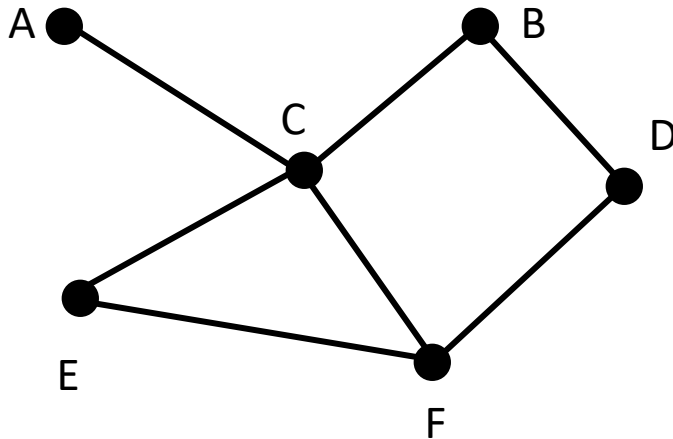
Walk Example

Consider the following graph representing the footpaths in a park:



Walk Definitions

A *walk* in a graph G is a sequence of vertices v_1, v_2, \dots, v_n where for each i , v_i and v_{i+1} are connected by an edge.



Example:

A,C,B,D,F,E,C,A

Types of Walks

A walk whose edges are distinct is called a *trail*.

A walk whose vertices are distinct is called a *path*.

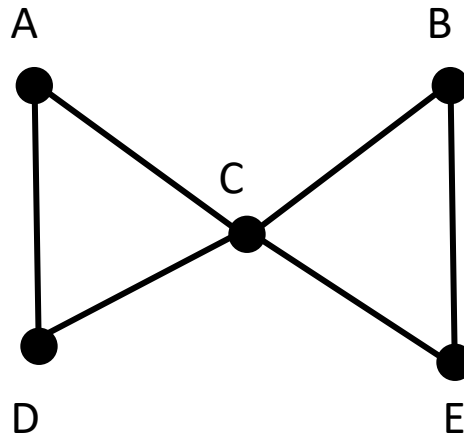
A *circuit* is a trail that starts and ends at the same vertex.

A *cycle* is a path plus an additional edge connecting the ends.

Question: Types of Walk

The sequence ACBECDA in the graph below is which of the following?

- A) Walk
- B) Path
- C) Trail
- D) Circuit
- E) Cycle



The Importance of Paths and Cycles

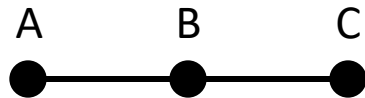
Lemma (Theorem 1.2): In a graph G every walk from vertex u to vertex v (a u - v walk) contains a u - v path (by removing some of the edges).

Similarly, every circuit contains a cycle.

Induction on Length

The *length* of a walk is the number of edges in that walk.

For example, the walk ABCBA below is length 4.

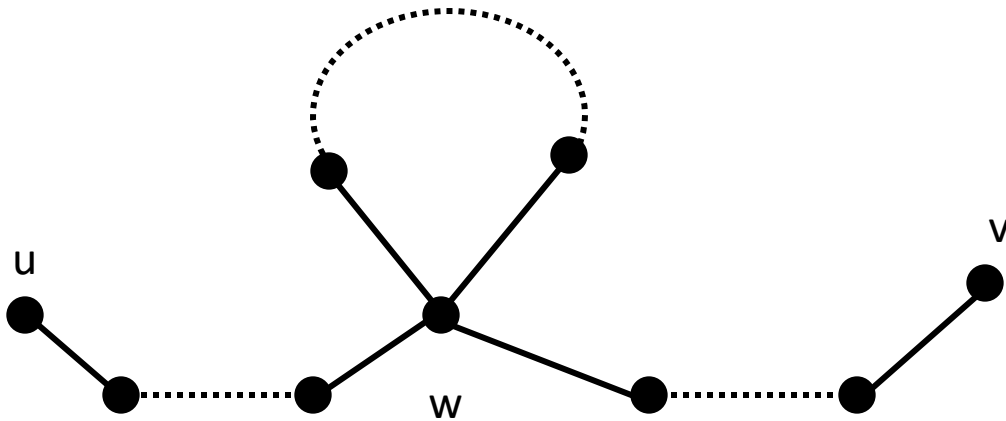


We prove our theorem by strong induction on the length of the walk/circuit.

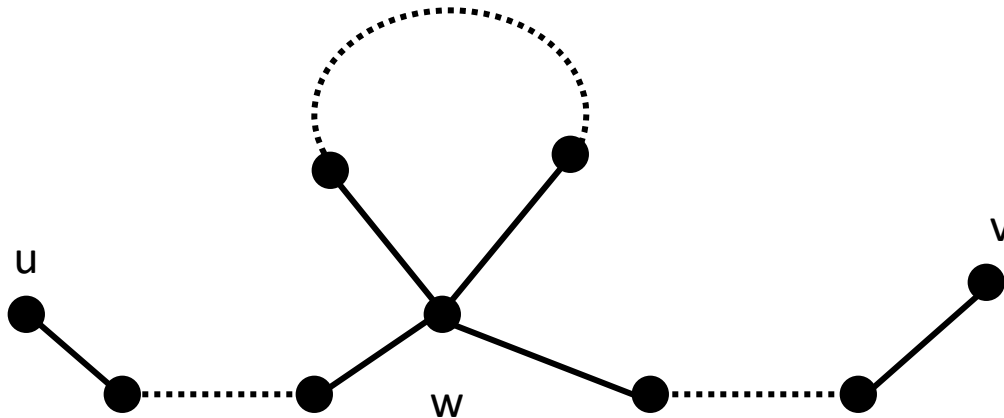
Proof I

Assume our walk W is *not* a path, but every shorter walk contains one.

Since it is not a path, it must repeat some vertex, w .



Proof II



- Walk must contain loop from w to itself.
- Remove loop to get shorter u - v walk.
- Inductive hypothesis implies that shorter walk contains a path.
- Similar argument for cycles.

Question: Maximum Distance

In a graph G the *distance* between two vertices u and v is the shortest length of a u - v walk.

What is the greatest possible distance between two vertices in a graph with n total vertices?

A) 1

B) $n-1$ A path can have at most
 $n-1$ edges

C) n

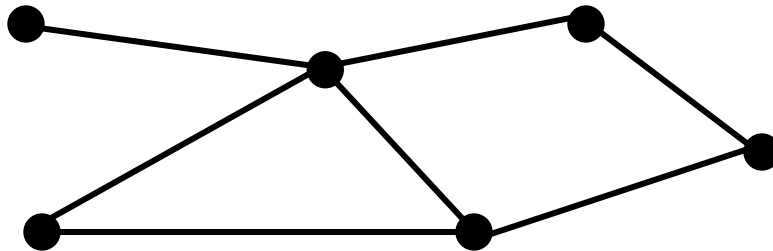
D) $n(n-1)/2$

E) Could be arbitrarily large

Connectivity

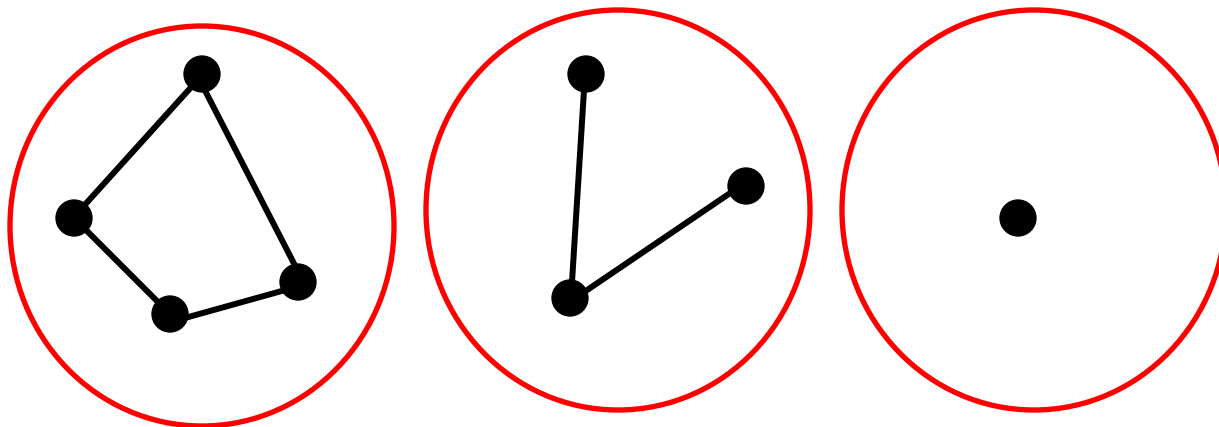
It is often useful to talk about how easy it is to “get around” a graph. At the most basic level we can ask is it possible to get around at all.

A graph G is *connected* if for any two vertices, u and v there is a u - v path in G .



Connected Components

Theorem: Any graph G can be uniquely partitioned into *connected components*, where each component is a connected subgraph and no two components have any edges between them.



Question: Connected Components

How many connected components does the graph below have?

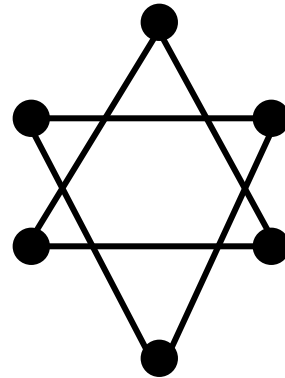
A) 0

B) 1

C) 2

D) 3

E) 4



Proof I

For vertices u and v in G say u and v are connected (or $u \sim v$) if there's a u - v path (equivalently a u - v walk) in G .

Want to show that this is an *equivalence relation*. Namely,

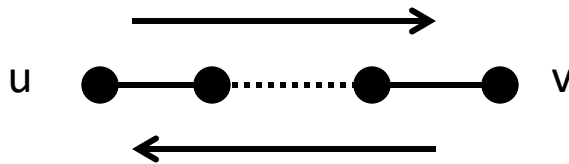
- 1) $v \sim v$ for all v .
- 2) If $u \sim v$ then $v \sim u$.
- 3) If $u \sim v$ and $v \sim w$ then $u \sim w$.

Equivalence Relation

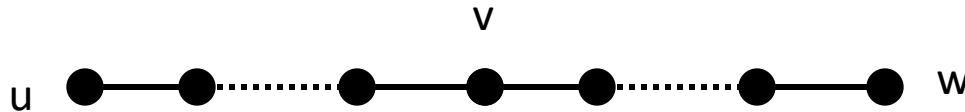
1) Trivial path from v to itself.



2) Reverse path from u to v , to get path from v to u .



3) Combine u - v walk with v - w walk to get u - w walk.



Proof II

Define the component of a vertex v to be the subgraph induced by the set of vertices u with $u \sim v$.

Note:

- 1) Each vertex in its own component.
- 2) If two components overlap, they are the same
 - A) Suppose $\text{comp}(v)$ and $\text{comp}(u)$ both contain w
 - B) $v \sim w$, and $u \sim w$ so $v \sim u$
 - C) For x in $\text{comp}(v)$, $v \sim x$ so $u \sim x$ so x in $\text{comp}(u)$