Announcements

- Please read final exam instructions

Last Time

- Ramsey Theory

Ramsey's Theorem

Theorem: For any positive integers p and q there exists a number N so that for any $n \geq N$ and any red-blue coloring of the edges of a K_n, there is either a red K_p or a blue K_q.

Definition: The smallest such number N is called the *Ramsey Number*, R(p,q).

$R(p,q) \leq R(p-1,q) + R(p,q-1)$

Today

- Approximating the size of Ramsey numbers
- Graph Ramsey Numbers

Computing Ramsey Numbers is Hard

Computing Ramsey numbers is notoriously hard, which is why so few of them are known. The problem is that even if you have a coloring determining whether it has monochromatic subgraphs is already difficult. Finding the best colorings is even harder. And as far as anyone can tell there doesn't seem to be an easy formula for Ramsey numbers.

Bounds on Ramsey Numbers

We will focus on proving bound to at least get an idea of how big Ramsey numbers are.

Upper Bound

Theorem: $R(p,q) \leq 2^{p+q}$.

Proof: By induction on p+q.

- If p=1 or q=1, $R(p,q) = 1 < 2^{p+q}$.
- Assume the inequality holds for smaller p+q.
  - $R(p,q) \leq R(p-1,q)+R(p,q-1)$
    $\leq 2^{p+q-1}+2^{p+q-1} \leq 2^{p+q}$.

Lower Bound

Theorem (1.66): If $n \geq 3$, $R(n,n) \geq 2^{n/2}$.

Note: $R(p,q) \geq 2^{\min(p,q)/2}$.

Note 2: Combined with the upper bound, this says that symmetric Ramsey numbers are exponentially large.

Note 3: Bound is hard. We want to find a coloring with no monochromatic subgraph. But actual constructions tend to produce patters. How do we avoid them?

Random Construction

Color the edges of a K_N randomly. On average how many monochromatic K_ns?

- $\approx N^n$ many collections of n vertices.
- Each has a $\approx 2^{-n(n-1)/2}$ probability of being monochromatic
- Average number of monochromatic Kns is roughly $N^n/(2^{n(n-1)/2}) \approx [N/2^{(n-1)/2}]^n$.
- If N much smaller than $2^{n/2}$, this is less than 1, so *some* coloring must have none.

Graph Ramsey Numbers

Traditional Ramsey Numbers look for complete subgraphs, but we can consider other kinds instead.

Definition: For graphs G and H, we define the graph Ramsey number R(G,H) to be the minimum n so that any red-blue coloring of K_n has either a red copy of G or a blue copy of H.

Finiteness

Note that G and H are contained in complete graphs, so this is finite.

Theorem (1.67):
$R(G,H) \leq R(|V_G|, |V_H|)$

Proof: Let $m = R(|V_G|, |V_H|)$. Any red-blue coloring of K_m has either a monochromatic complete red graph on $|V_G|$ or monochromatic blue complete graph on $|V_H|$. These contain a red copy of G or blue copy of H.

Example

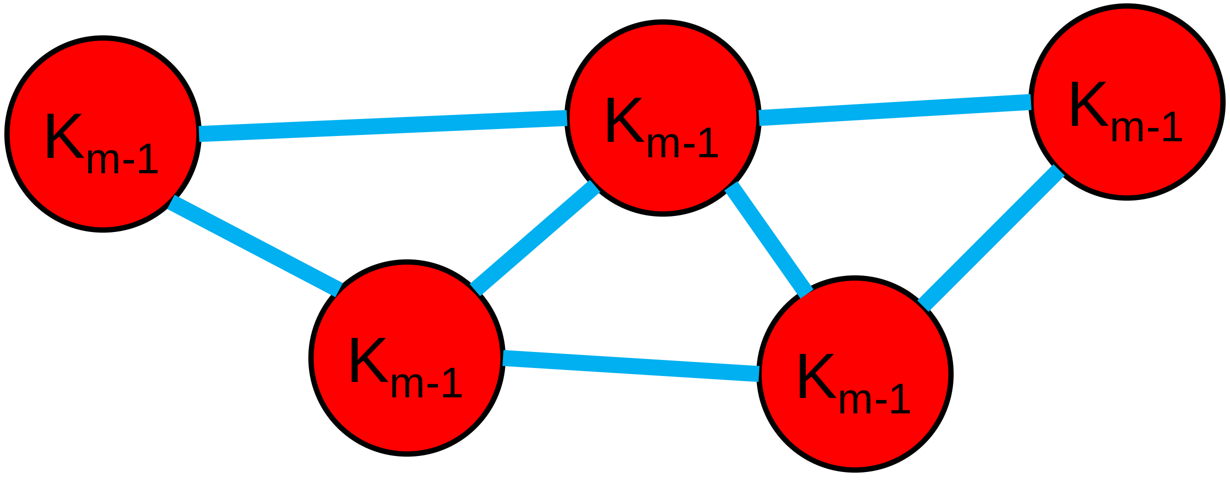
Theorem (1.70): If m and n are integers with m-1 dividing n-1 and T_m is a tree with m vertices then

$$R(T_m, K_{1,n}) = m+n-1.$$

Lower Bound

- Need a coloring of K_{m+n-2} without a red T_m or blue K_n.
- Note: m-1 divides m+n-2 = (m-1)+(n-1).

Coloring



- Red K_{m-1}s connected by blue edges.
- No Red T_m: All but CCs size m-1.
- No Blue $K_{1,n}$: Each vertex has blue degree (m+n-3) – (m-2) = n-1.

Upper Bound

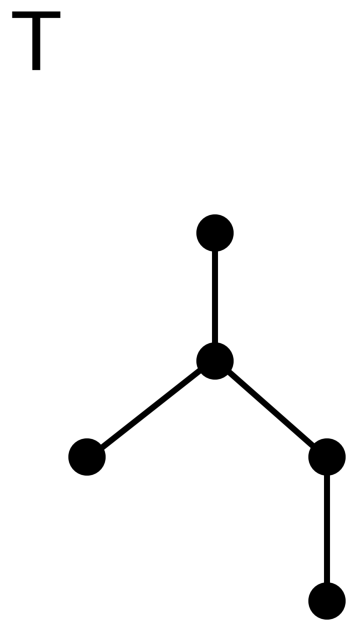
- Need to show that any red-blue coloring of a K_{n+m-1} has either a red T_m or a blue $K_{1,n}$.
- If any vertex has n or more blue edges, have blue $K_{1,n}$.
- Otherwise, consider G_r, graph of red edges.
  - Note that $\delta(G_r) \geq m-1$.

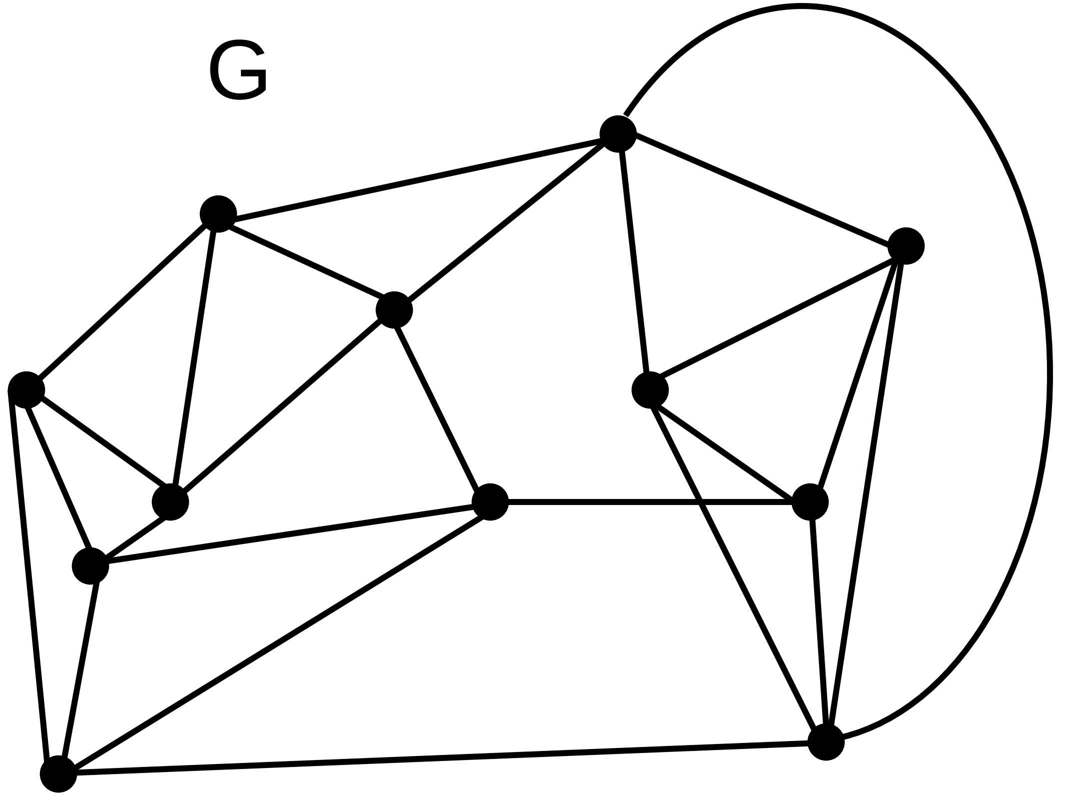
Lemma

Lemma (1.16): Let T be any tree on k vertices and G a graph with $\delta(G) \geq k-1$. Then G contains a copy of T.

Apply to G_r and T_m to get final result.

Idea: Build T One Vertex at a Time





Proof by Induction on k

- Base case: k = 1
  - Can embed single point
- Assume can embed any tree on k-1 vertices.
- Let v be a leaf of T. Removing v and edge (u,v) gives T'.
- By IH, embed T' in G.
- Need new neighbor of u to be v.
- u has k-1 neighbors, only k-2 are used.