Announcements

• Please read final exam instructions
Last Time

• Ramsey Theory
Ramsey’s Theorem

**Theorem:** For any positive integers $p$ and $q$ there exists a number $N$ so that for any $n \geq N$ and any red-blue coloring of the edges of a $K_n$, there is either a red $K_p$ or a blue $K_q$.

**Definition:** The smallest such number $N$ is called the *Ramsey Number*, $R(p, q)$.

$R(p, q) \leq R(p-1, q) + R(p, q-1)$
Today

• Approximating the size of Ramsey numbers
• Graph Ramsey Numbers
Computing Ramsey Numbers is Hard

Computing Ramsey numbers is notoriously hard, which is why so few of them are known. The problem is that even if you have a coloring determining whether it has monochromatic subgraphs is already difficult. Finding the best colorings is even harder. And as far as anyone can tell there doesn’t seem to be an easy formula for Ramsey numbers.
Bounds on Ramsey Numbers

We will focus on proving bound to at least get an idea of how big Ramsey numbers are.
**Theorem:** $R(p, q) \leq 2^{p+q}$.

**Proof:** By induction on $p+q$.

- If $p=1$ or $q=1$, $R(p, q) = 1 < 2^{p+q}$.
- Assume the inequality holds for smaller $p+q$.
  
  - $R(p, q) \leq R(p-1, q) + R(p, q-1) 
  \leq 2^{p+q-1} + 2^{p+q-1} \leq 2^{p+q}$. 

Lower Bound

**Theorem (1.66):** If $n \geq 3$, $R(n,n) \geq 2^{n/2}$.

**Note:** $R(p,q) \geq 2^{\min(p,q)/2}$.

**Note 2:** Combined with the upper bound, this says that symmetric Ramsey numbers are exponentially large.

**Note 3:** Bound is hard. We want to find a coloring with no monochromatic subgraph. But actual constructions tend to produce patterns. How do we avoid them?
Random Construction

Color the edges of a $K_N$ randomly. On average how many monochromatic $K_n$s?

- $\approx N^n$ many collections of $n$ vertices.
- Each has a $\approx 2^{-n(n-1)/2}$ probability of being monochromatic
- Average number of monochromatic $K_n$s is roughly $N^n/(2^{n(n-1)/2}) \approx [N/2^{(n-1)/2}]^n$.
- If $N$ much smaller than $2^{n/2}$, this is less than 1, so some coloring must have none.
Graph Ramsey Numbers

Traditional Ramsey Numbers look for complete subgraphs, but we can consider other kinds instead.

**Definition:** For graphs $G$ and $H$, we define the graph Ramsey number $R(G,H)$ to be the minimum $n$ so that any red-blue coloring of $K_n$ has either a red copy of $G$ or a blue copy of $H$. 
Finiteness

Note that G and H are contained in complete graphs, so this is finite.

**Theorem (1.67):**
\[ R(G,H) \leq R(|V_G|, |V_H|) \]

**Proof:** Let \( m = R(|V_G|, |V_H|) \). Any red-blue coloring of \( K_m \) has either a monochromatic complete red graph on \( |V_G| \) or monochromatic blue complete graph on \( |V_H| \). These contain a red copy of G or blue copy of H.
Example

**Theorem (1.70):** If $m$ and $n$ are integers with $m-1$ dividing $n-1$ and $T_m$ is a tree with $m$ vertices then

$$R(T_m,K_{1,n}) = m+n-1.$$
Lower Bound

• Need a coloring of $K_{m+n-2}$ without a red $T_m$ or blue $K_n$.
• Note: $m-1$ divides $m+n-2 = (m-1)+(n-1)$. 
• Red $K_{m-1}$s connected by blue edges.
• No Red $T_m$: All but CCs size $m-1$.
• No Blue $K_{1,n}$: Each vertex has blue degree 
  $(m+n-3) - (m-2) = n-1.$
Upper Bound

• Need to show that any red-blue coloring of a $K_{n+m-1}$ has either a red $T_m$ or a blue $K_{1,n}$.
• If any vertex has $n$ or more blue edges, have blue $K_{1,n}$.
• Otherwise, consider $G_r$, graph of red edges.
  – Note that $\delta(G_r) \geq m-1$. 
Lemma

**Lemma (1.16):** Let $T$ be any tree on $k$ vertices and $G$ a graph with $\delta(G) \geq k-1$. Then $G$ contains a copy of $T$.

Apply to $G_r$ and $T_m$ to get final result.
Idea: Build T One Vertex at a Time
Proof by Induction on $k$

• Base case: $k = 1$
  – Can embed single point
• Assume can embed any tree on $k-1$ vertices.
• Let $v$ be a leaf of $T$. Removing $v$ and edge $(u,v)$ gives $T'$.
• By IH, embed $T'$ in $G$.
• Need new neighbor of $u$ to be $v$.
• $u$ has $k-1$ neighbors, only $k-2$ are used.