Announcements

- If you cannot conveniently take the final at the regular time, please let me know by the end of the day.
- Final exam is optional (your score on the final will be the maximum of your score on the exam and the average of your midterm scores).

Ramsey Theory (Ch 1.8)

- Introduction
- Definitions
- Existence of Ramsey Numbers
- Derivation of Small Ramsey Numbers
- Lower Bounds
- Other Ramsey Problems

A Party

Suppose you are at a party with at least 6 people in it. Show that there are either 3 people who all know each other or 3 people who are all strangers to each other.

Proof

Pick someone at the party, say Alice.

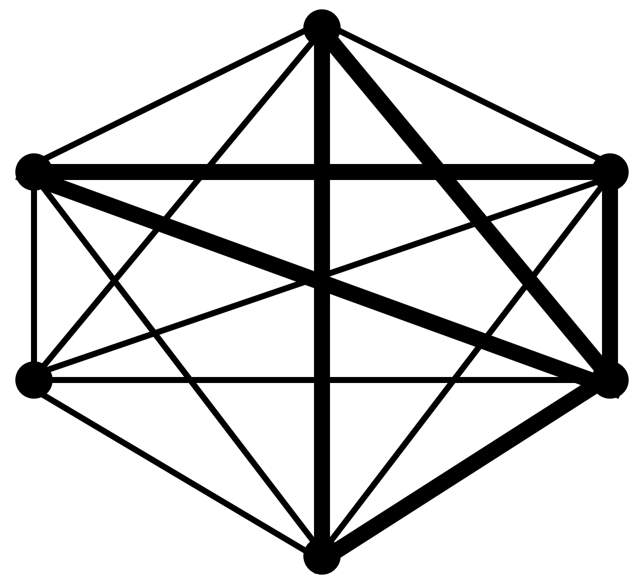
There are 5 other people so, she either knows at least 3 of them or doesn’t know at least 3.

Say she knows Bob, Charles and Diane.

- If any two of Bob, Charles or Diane know each other, we have three mutual friends.
- If no two know each other, we have three mutual strangers.

Graph Theory

- 6 people
- Every pair either friends or strangers
- Three friends or three strangers.



Friends
Strangers

Theorem

Theorem: Given any coloring of the edges of a K_6 with two colors, there is always a monochromatic K_3 subgraph.

Ramsey's Theorem

Theorem: For any positive integers p and q there exists a number N so that for any $n \geq N$ and any red-blue coloring of the edges of a K_n, there is either a red K_p or a blue K_q.

Definition: The smallest such number N is called the *Ramsey Number*, R(p,q).

Proof I

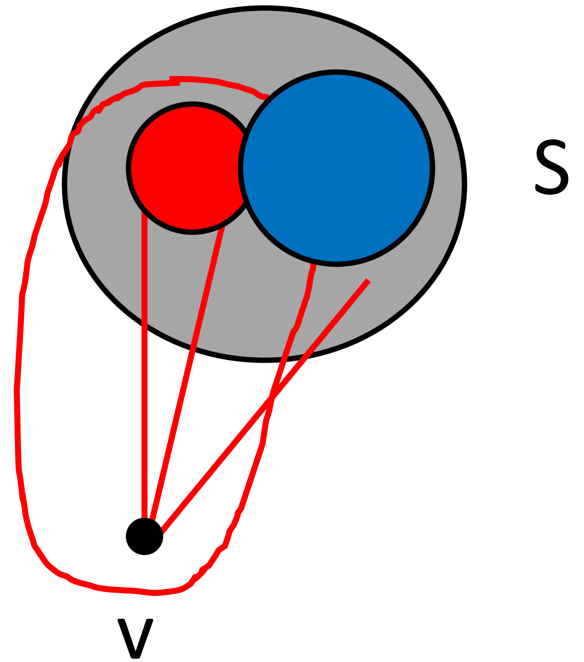
- We proceed by induction on p+q.
- Base case: If p = 1 or q = 1, then any coloring of a K_1, has a red/blue K_1. So R(1,q) = R(p,1) = 1.
- Inductive step: Assume R(p',q') is finite for all p'+q' < p+q.
- For p, q > 1, we will show not only that R(p,q) is finite, but that
  $R(p,q) \leq R(p-1,q)+R(p,q-1)$.

Proof II

- Take $n \geq R(p-1,q)+R(p,q-1)$. Color K_n.
- Consider vertex v, has at least $R(p-1,q)+R(p,q-1)-1$ edges out of it.
- Either has $R(p-1,q)$ red edges or $R(p,q-1)$ blue edges.
- WLOG v has $R(p-1,q)$ red edges to other vertices S.

Proof III

- Have v, S
  - All edge v to S red.
  - |S| ≥ R(p-1,q).
- IH: Color of S has either:
  - Red K_{p-1}.
    - v + clique is red K_p
  - Blue K_q
    - Gives blue K_q
- Either way we're done.



Ramsey Theory

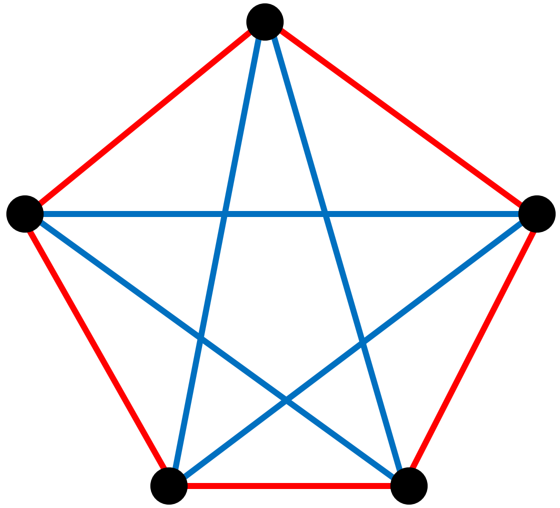
Ramsey Theory more generally studies these kinds of patterns and when certain types of structures *must* exist within sufficiently much noise.

Computing Ramsey Numbers

- $R(p,q) = R(q,p)$ by symmetry.
- $R(1,n) = 1$ as discussed.
- $R(2,n) = n$
  - Any red edge gives red K_2.
  - All blue edges gives blue K_n.
  - But coloring a K_{n-1} all blue gives neither.

R(3,3) = 6

- $R(3,3) \leq R(2,3)+R(3,2) = 3+3 = 6$.



$R(3,4) = 9$

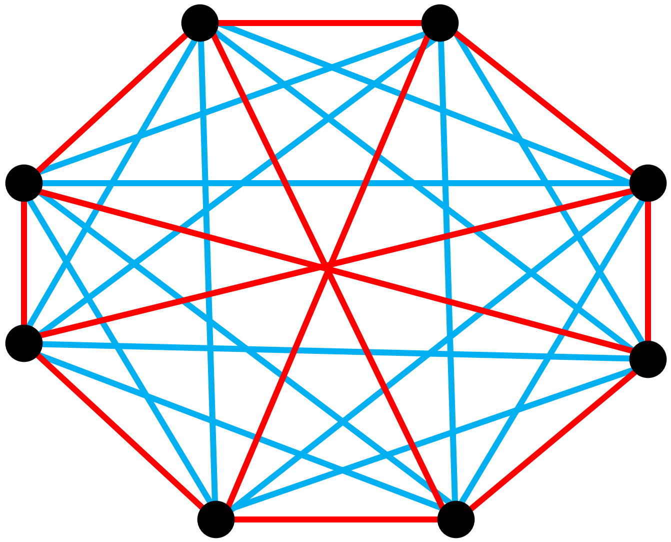
Upper bound: Show that any red-blue coloring of a K_9 has either a red K_3 or blue K_4.

Note: Bound we know $R(3,3)+R(2,4) = 10$.

- By previous argument, if any vertex has 6 blue edges or 4 red edges, done.
- Otherwise, each vertex must have 5 blue, 3 red edges.
- Violates Handshake Lemma. Total number of red edges = 9*3/2 = 13.5.

R(3,4) = 9

Lower bound: Need a coloring of a K_8 with no red K_3 or blue K_4.



$R(4,4) = 18$

- $R(4,4) \leq R(3,4) + R(4,3) = 9 + 9 = 18$
- Lower Bound:
  – Arrange 17 vertices in a circle
  – Draw a red edge between two if they are separated by 1, 2, 4, or 8 steps
  – Other edges are blue
  – Symmetry makes analysis relatively easy

Other Known Ramsey Numbers

- R(3,5) = 14
- R(3,6) = 18
- R(3,7) = 23
- R(3,8) = 28
- R(3,9) = 36
- R(4,5) = 25

Some Additional Bounds

- $35 \leq R(4,6) \leq 41$
- $43 \leq R(5,5) \leq 49$
- $58 \leq R(5,6) \leq 87$
- $102 \leq R(6,6) \leq 165$