Announcements

- No more Homeworks!
- If you need an alternative time for the final, please let me know by Wednesday.
- All regrade requests due before final

Last Time

- Perfect Matchings in General Graphs

Today

- Tutte’s Theorem
- Introduction to Ramsey Theory

In General

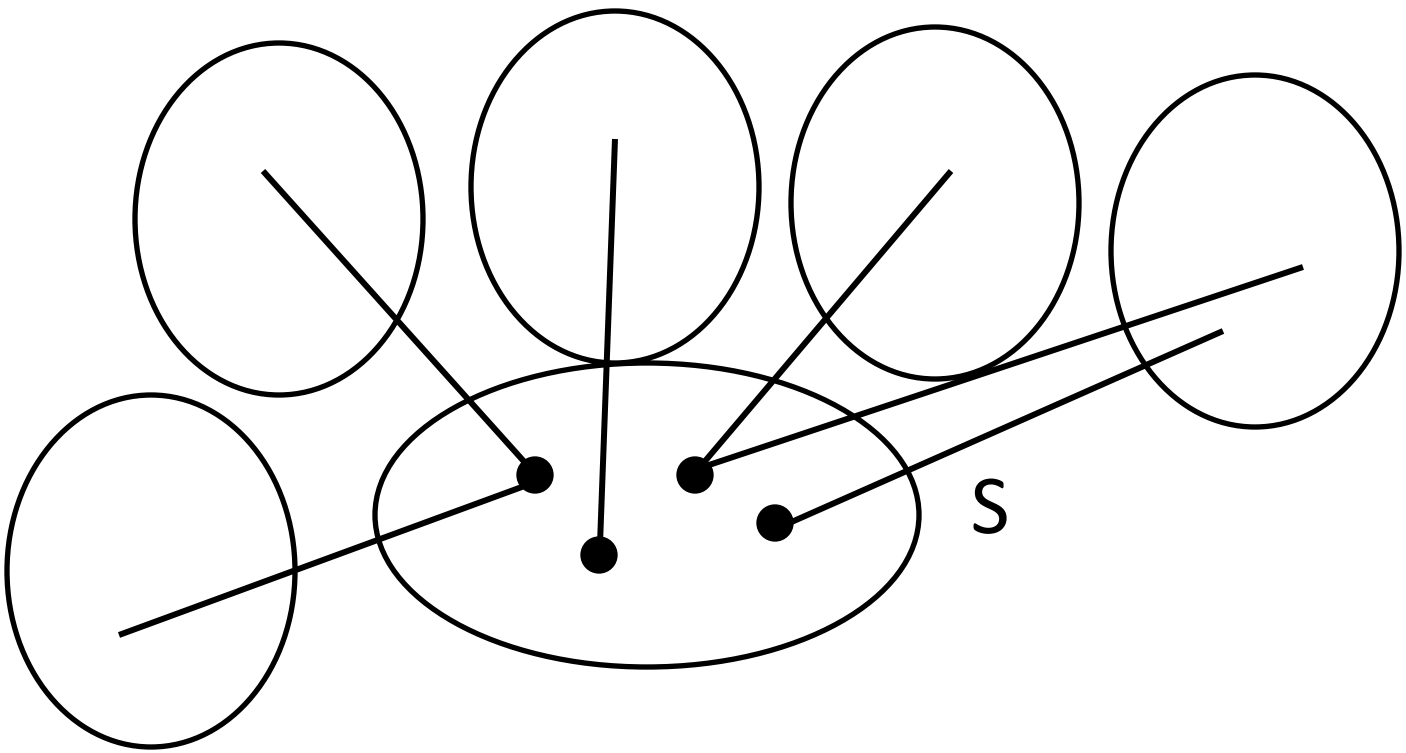
Odd vs. Even is important.

Definition: For a graph G let $\Omega(G)$ denote the number of connected components of G with an odd number of vertices.

Lemma: If there is a set S of vertices of G with $|S| < \Omega(G-S)$, then G has no perfect matching.

Proof

Each of the $\Omega(G-S)$ odd components would need at least one edge to an element of S, but there are not enough to go around.



Tutte’s Theorem

Surpsingly, this is the only thing that can go wrong.

<u>Theorem (1.59):</u> If G is a finite graph so that for every set S of vertices $|S| \geq \Omega(G-S)$, then G has a perfect matching.

Proof Strategy

Need to show that if $|S| \geq \Omega(G-S)$ for all S then G has a perfect matching.

- Use induction on $|V|$.
- Find maximal S so that $|S| = \Omega(G-S)$.
- Use Hall's Theorem to find matching on S.
- Induct.

A Lemma

Lemma: If G has an even number of vertices, then S-Ω(G-S) is always even.

Note: If G has an odd number of vertices, then for S = ∅, |S| < Ω(G-S).

Proof: The number of vertices of G equals |S| plus the sum of the sizes of the connected components of G-S. If a collection of numbers adds to an even number, it must contain an even number of odd numbers.

Proof

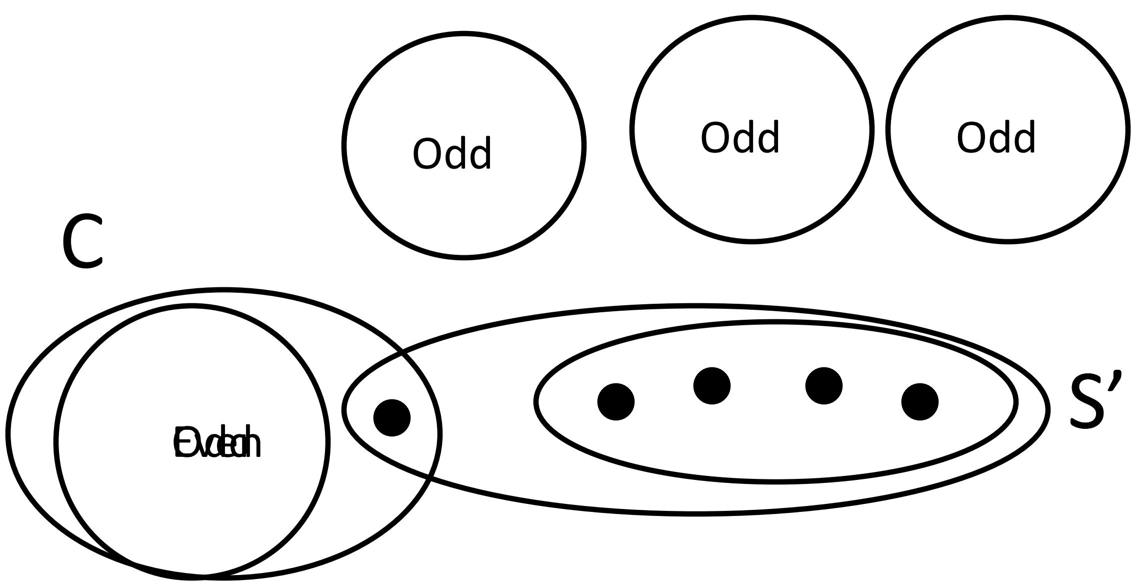
- Use induction on $|V|$.
  - Base Case: $|V|=0$, empty pairing.
  - Assume that Tutte's Theorem holds for all smaller graphs.
- For a G satisfying our hypothesis find a *maximal* set S with $|S| = \Omega(G\text{-}S)$.
  - Since $|V|$ is even taking S = {single point} gives $|S| = 1$, $\Omega(G\text{-}S) \geq 1$.

Claim 1

Every component of G-S has an odd number of vertices.

If some component C was even, taking $S' = S \cup \{v\}$ for $v \in C$ also has $|S'| = \Omega(G-S')$

S isn't maximal.



Claim 2

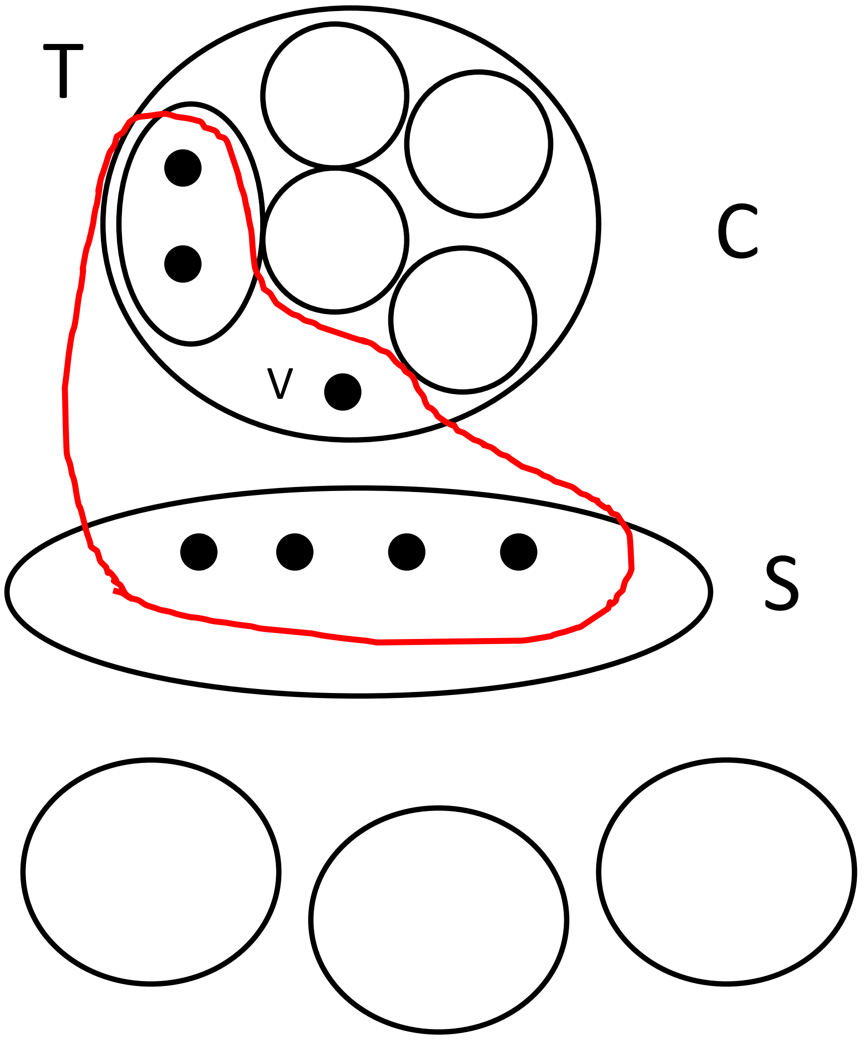
For each component C of G-S and each $v \in C$ there is a perfect matching of C-v.

<u>**Proof Idea:**</u>

- Inductive hypothesis on C-v.
- Use parity lemma.
- Use maximality of S.

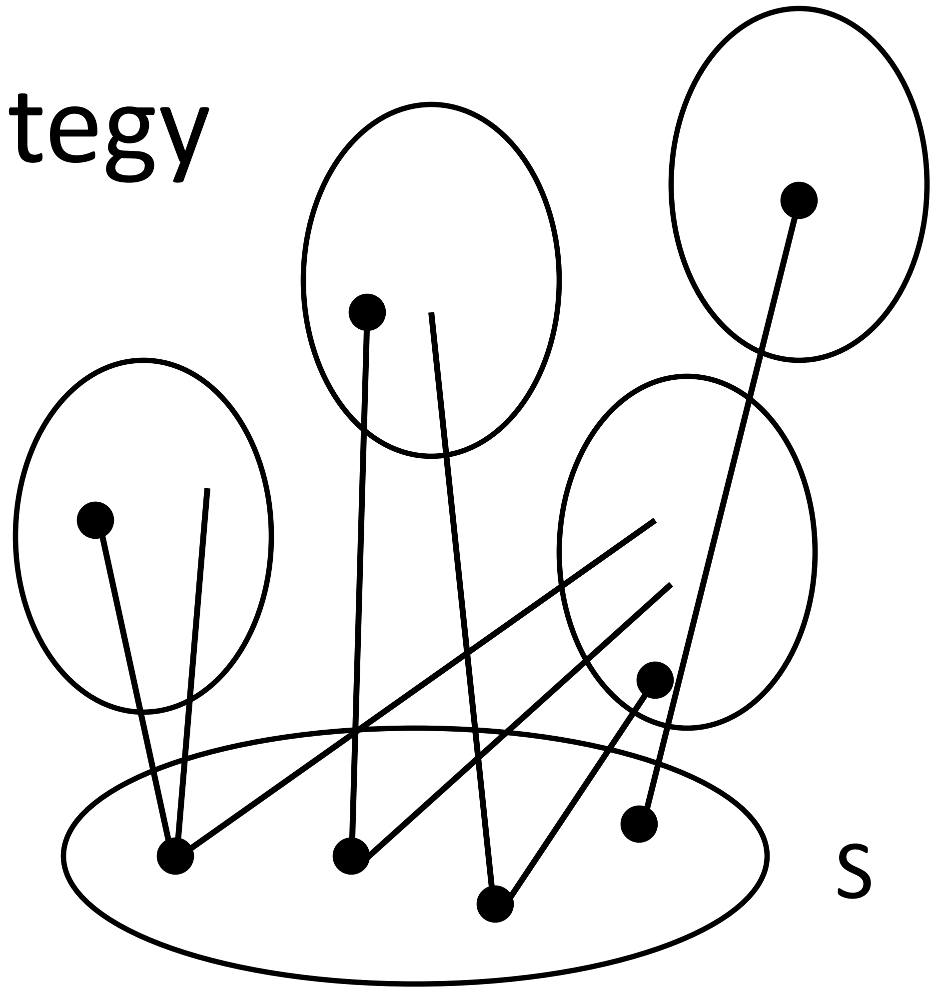
Proof of Claim 2

- By IH, have matching of C-v, unless T so that |T| < Ω(C-T-v).
- Parity lemma: |T| ≤ Ω(C-T-v)-2.
- Ω(G-(S UT U{v})) = Ω(G-S)-1+Ω(C-T-v) ≥ |S UT U{v}|
- Contradicts maximality.



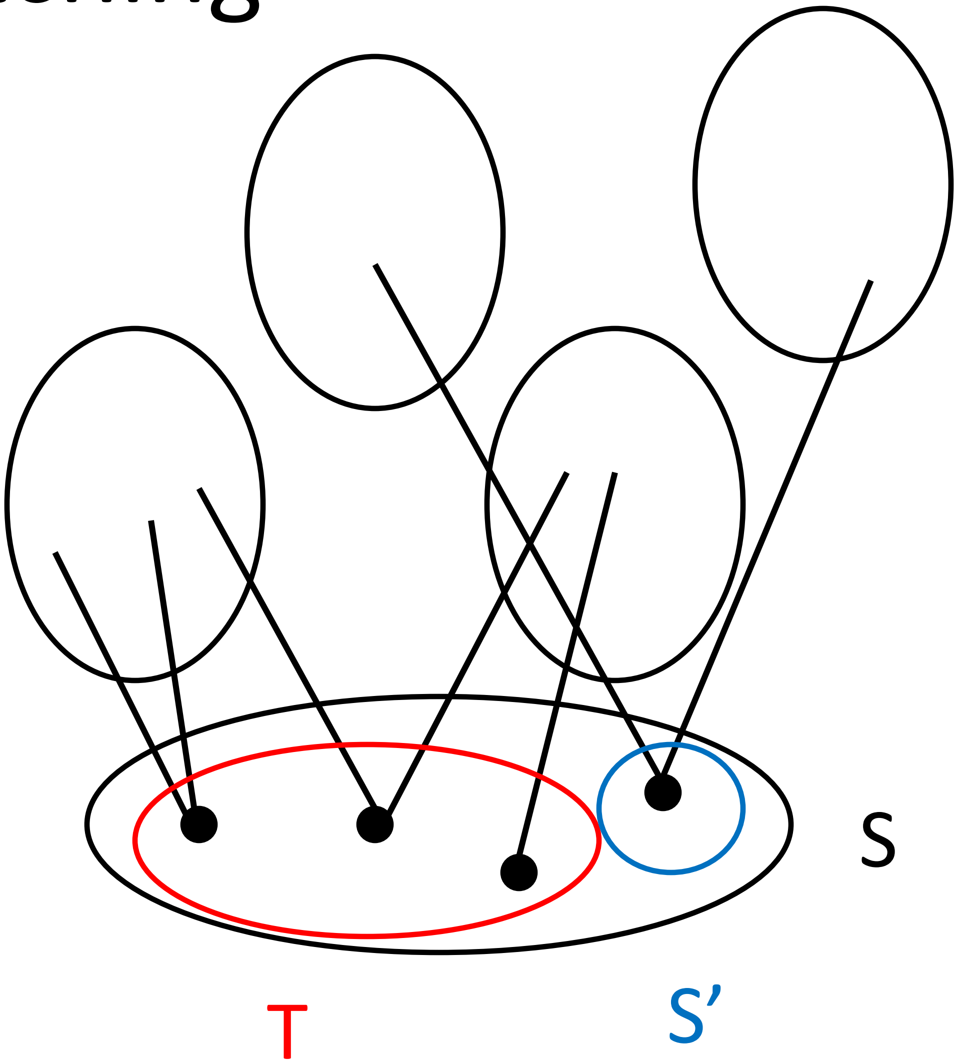
Strategy

- Have |S| = Ω(G-S).
- Want: matching between S and the components of G-S
- If we had this, remaining vertices could be matched.
- This is a bipartite matching problem!



Matching

- Hall's Theorem
  - Matching unless $T \subseteq S$ adjacent to $<|T|$ components.
- Let $S' = S-T$.
- $\Omega(G-S') \geq \Omega(G-S)-|N(T)| = |S|-|N(T)| > |S|-|T| = |S'|$
- Contradiction!



Outline of Proof

- Find maximal S with $|S| = \Omega(G\text{-}S)$.
- Hall's Theorem ⇒ matching between elements of S and odd components.
- Use those edges in matching.
- Inductive hypothesis and maximality imply matching of C-v for each component C and matched vertex v.
- Combine to get full matching.

Question: Algorithm

Does the proof of Tutte's Theorem lend itself to an efficient algorithm to find a matching?

A) Yes
B) No

The proof relies on being able to find a maximal set S with $|S| = \Omega(G\text{-}S)$.

Application: Petersen's Theorem

Theorem (1.60): Any bridgeless, 3-regular graph has a perfect matching.

Proof Idea: Use Tutte's Theorem. Use 3-regular and bridgeless to assist counting.

Lemma 1

Lemma: If G is 3-regular, S a subset of the vertices and C an odd connected component of G-S, then there are an odd number of edges between C and S.

Proof: Apply Handshake lemma to the induced subgraph C.

Sum of degrees is even.

Sum of degrees is $3|C|$- #{outgoing edges}.

Lemma 2

Lemma: If G is a bridgeless graph, S a subset of vertices and C a connected component of G-S, then C cannot have exactly one edge to S.

Proof: That edge would be a bridge.

Combined

Corollary: If G is a 3-regular, bridgeless graph, S a subset of the vertices and C an odd connected component of G-S, then C has at least 3 edges to S.

Proof

- Let S be a subset of the vertices, let T be the set of odd components of G-S.
  - NTS: $|T| \leq |S|$
- Consider bipartite graph on S and T.
- $d(s) \leq 3$ for $s \in S$, $d(t) \geq 3$ for $t \in T$.
- $3|S| \geq \Sigma d(s) = \Sigma d(t) \geq 3|T|$
- $|S| \geq |T|$
- Tutte’s Theorem implies matching.