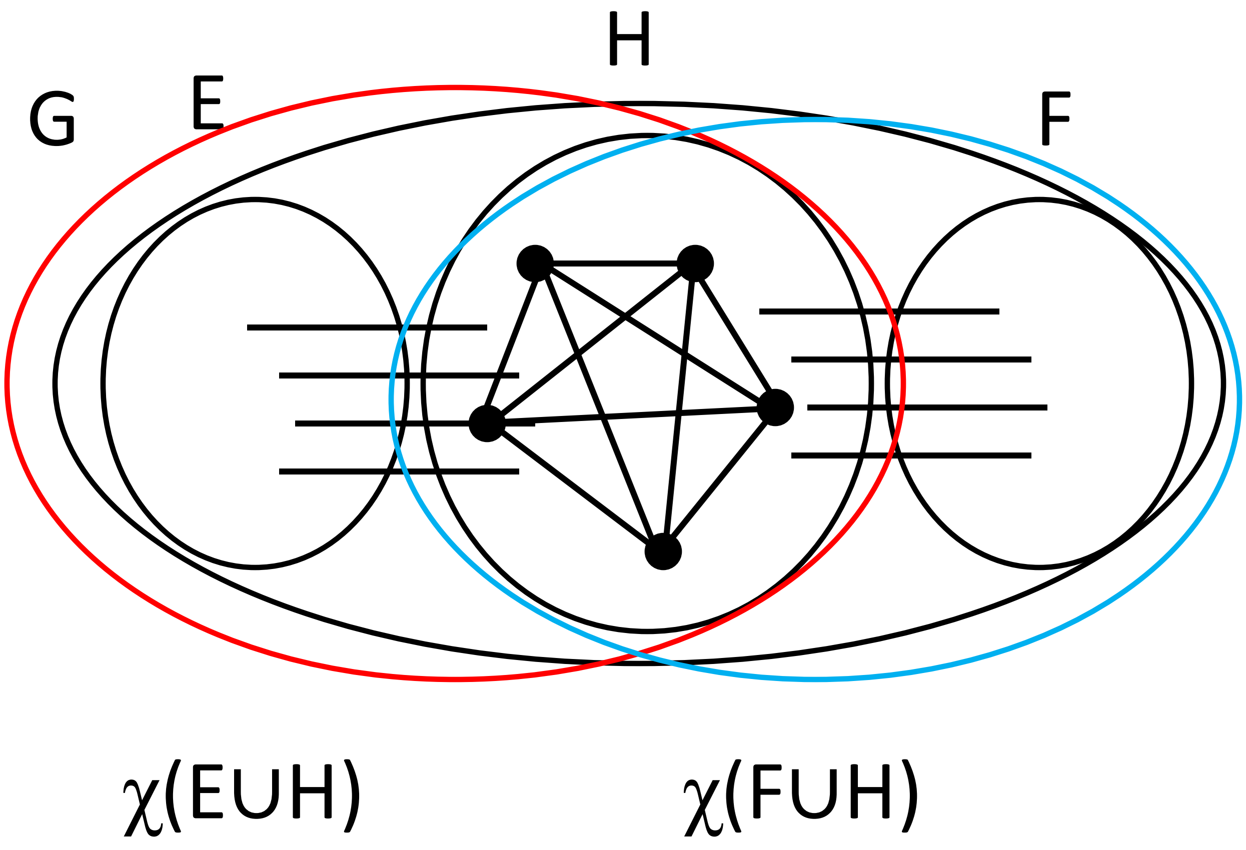
Announcements

- HW 6 (the last homework!) Due on Sunday
- Please Remember to fill out your CAPEs
- Exam 2 grades out (median 68)

Exam 2 Q4



Proof

- $\chi(G) \geq \max(\chi(E \cup H), \chi(F \cup H))$
  - Chromatic number of a graph at least the chromatic number of subgraph.
- $\chi(G) \leq \max(\chi(E \cup H), \chi(F \cup H))$
  - Color EUH with appropriate number of colors
  - Color FUH with appropriate number of colors
  - To combine, need colorings to agree on H
  - H complete so both give H *distinct* colors, by recoloring can make them agree

Notes

- Need to show both directions
- Need to use the fact that H is complete (false otherwise)

Today

- Introduction to Flows
- Maxflow-Mincut
- Applications

Evacuation Planning

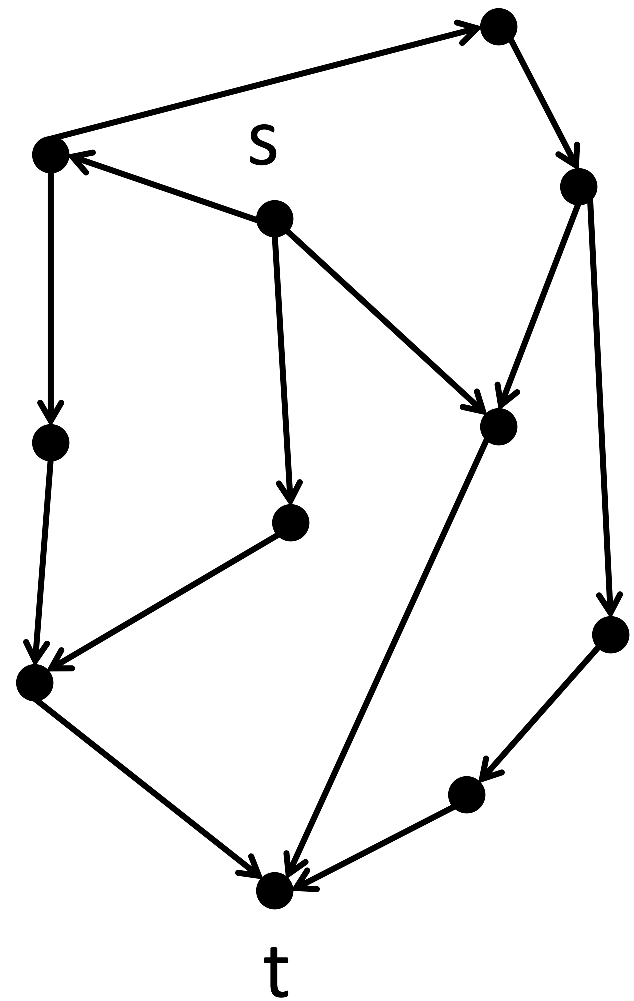
You are an emergency planner for a city. You need to plan evacuation routes to be used in the case of a disaster. You need to know two things:

1) Which routes to use
2) How quickly you can get people out of the city to the designated safe area

Note: You cannot just use all the roads out equally, some routes might get bottlenecked unless you plan carefully.

Formal Setup

- Have city (s) and evacuation site (t).
- Have (directed) graph of roads.
- Pick some roads to use.
- Need: For each vertex other than s,t, number of incoming roads used equals outgoing



Definitions

A *network* is a directed graph G with designated *source* and *sink* vertices s and t.

A *flow* is a subgraph of G so that for each vertex v other than s and t $d_{in}(v) = d_{out}(v)$.

The *size* of a flow is $d_{out}(s) – d_{in}(s)$.

Problems

- Given a network G, what is the largest size of a flow in the network?
- How do we find such a flow?
- How can we show that there isn’t a larger flow?

Question: Maxflow

What is the largest flow size in this network?

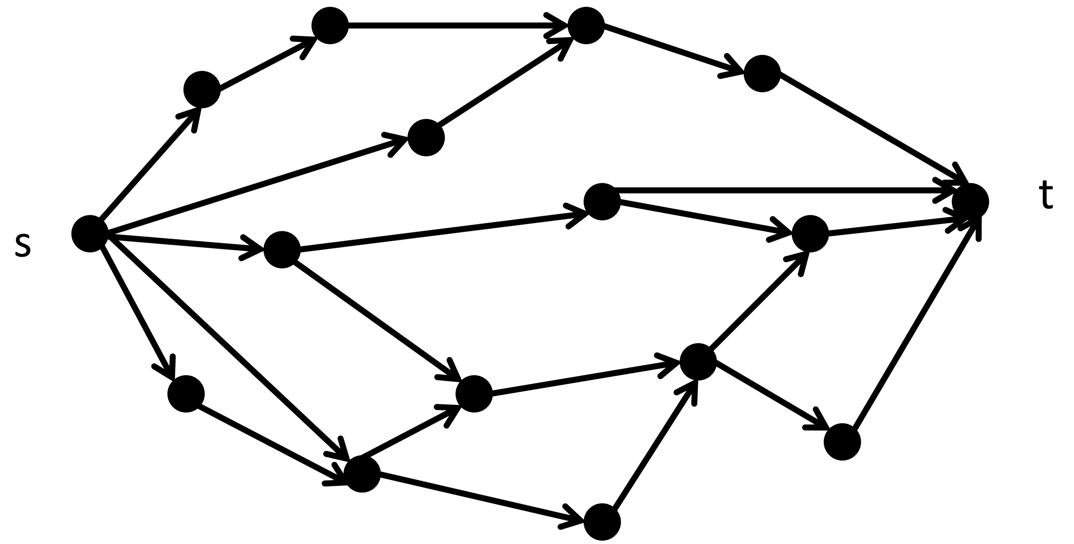
A) 1

B) 2

C) 3

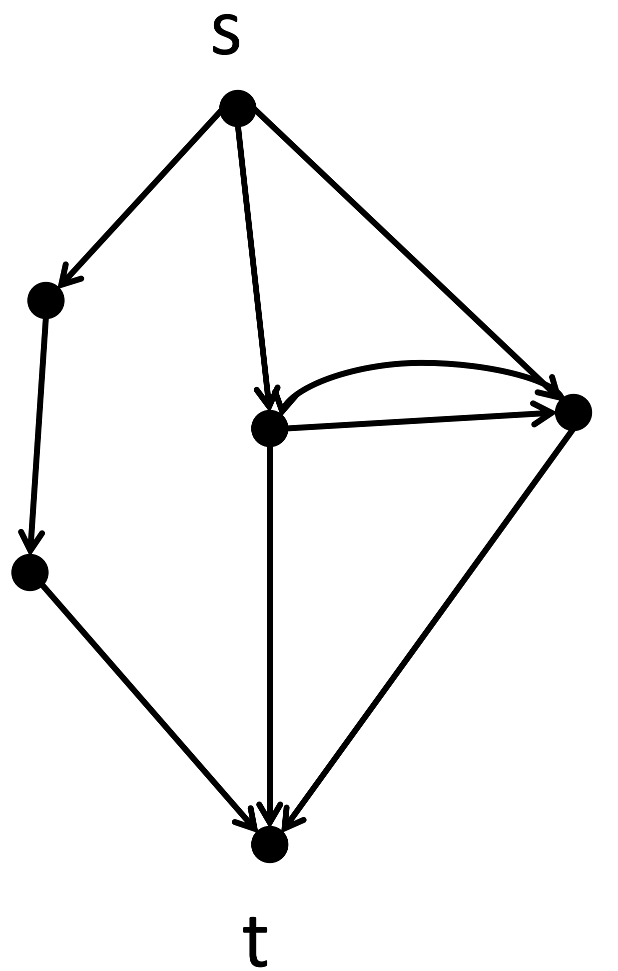
D) 4

E) 5



Creating Flow

- How do you find a flow?
  - Path from s to t
- Find more flow:
  - Add s-t paths
- Even better: these paths can "cancel" existing edges.



Augmenting Paths

Definition: Given a graph G and flow F an *augmenting path* is an s-t path that uses either edges of G unused by F in the forwards direction, or edges used by F in the backwards direction.

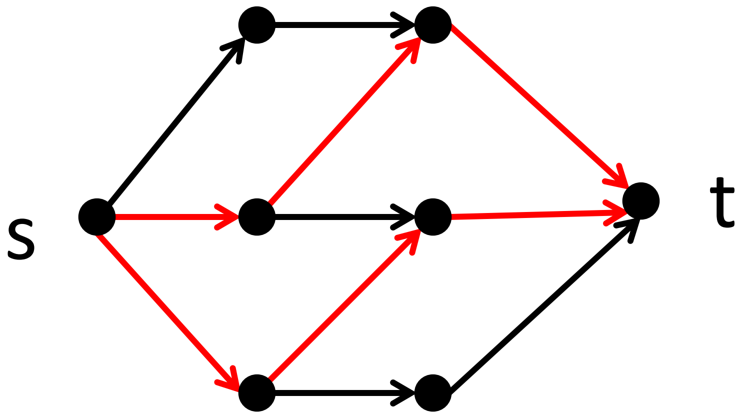
Lemma: Given an augmenting path, you can add it to F to get a path with 1 more unit of flow.

Question: Augmenting Path

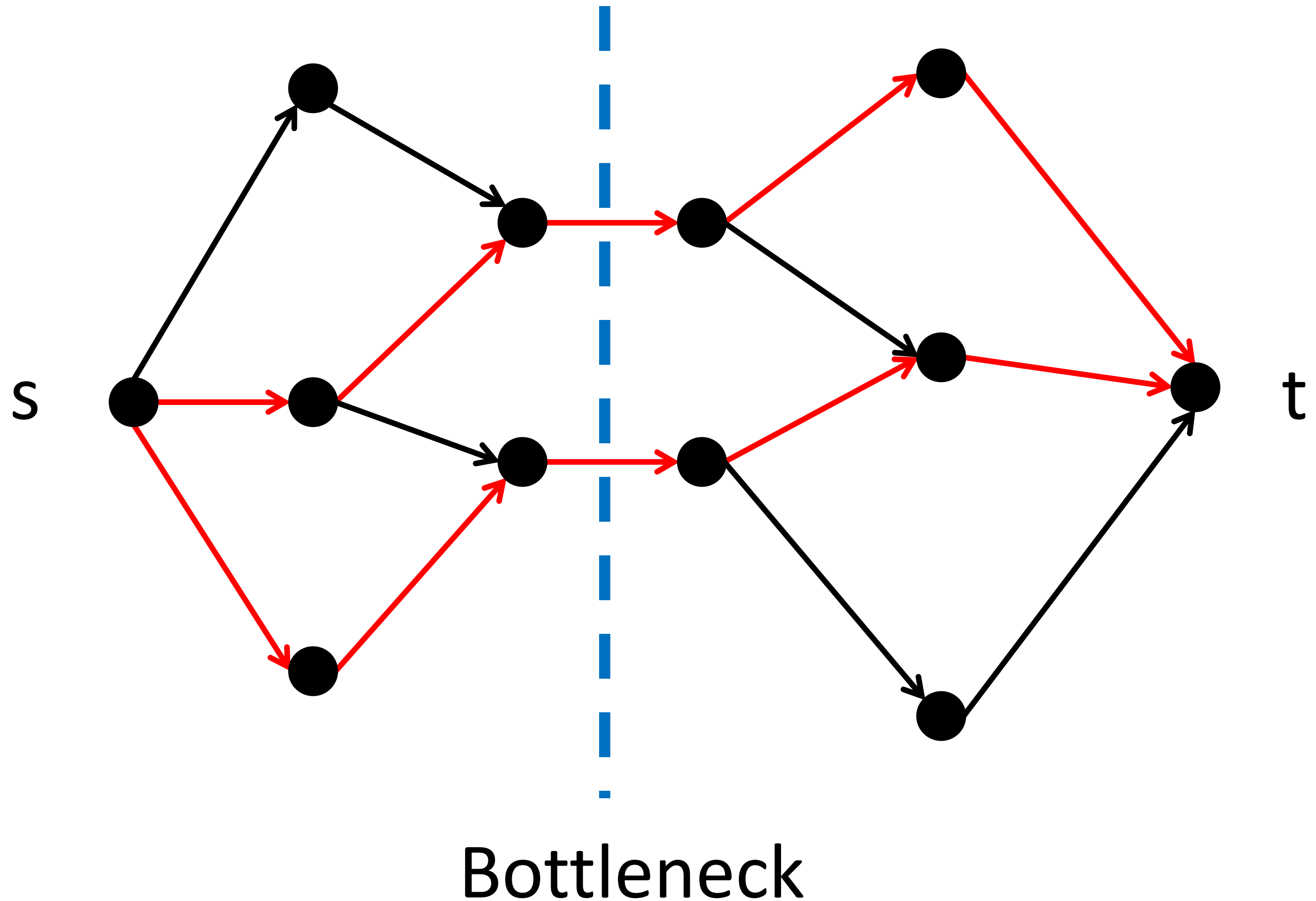
Does this flow have an augmenting path?

A) Yes

B) No



When Can't You Augment?



Cuts

Definition: A *cut* is a partition of the vertices into two sets S and T, which contain s and t, respectively.

The *size* of a cut is the total number of edges from vertices in S to vertices in T.

Question: Cut Size

What is the size of the cut below?

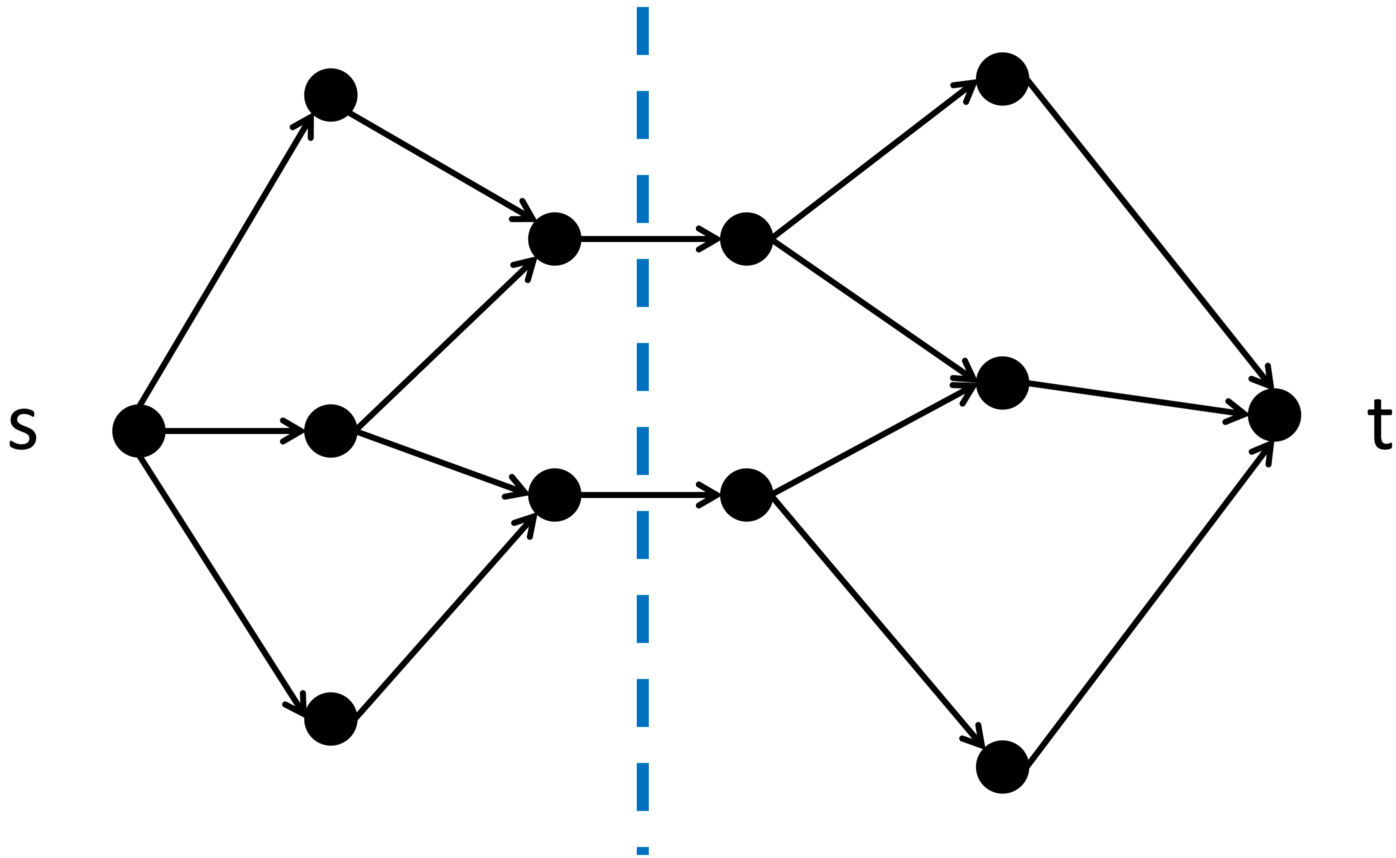
A) 1

B) 2

C) 3

D) 4

E) 5



Cuts and Flows

Lemma (V 8.3.1): For a network G a flow F and a cut (S,T) it is the case that

Size(F) = #{edges in F from S to T} - #{edges in F from T to S}

Remark: This says that the number of people leaving the city, is the number crossing into the next state (assuming that's where they are headed).

Proof

Consider the sum over all v in S of $d_{out}(v)-d_{in}(v)$.

On the one hand this is 0 except for v = s, where it is Size(F).

On the other hand, each edge contributes to one in degree and one out degree. This makes its total contribution 0 unless it crosses the cut. This gives 1 for each edge from S to T and -1 for each edge from T to S.

Note

If we take T = {t}, we find:
Size(F) = $d_{in}(t) - d_{out}(t)$.

The total flow out of s equals the total flow into t.

Maxflow-Mincut

Theorem (V. 8.3.2): For any network G the size of a maximum flow in G is the same as the size of a minimum cut.